Flat Surfaces of Finite Type in the 3-sphere

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Thesis submitted for the degree of Doctor of Philosophy

May 2014

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I, Alan McCarthy, certify that this thesis is my own work and I have not obtained a degree in this university or elsewhere on the basis of the work submitted in this thesis.

Alan McCarthy

For Bernie and Noel.

Acknowledgements

I would like to thank my supervisor Martin Kilian for his tireless encouragement and open door policy that meant I always felt welcomed and never felt stuck and for his unrelenting support which has made this PhD a joy to work with him on. I would also like to thank Tom Carroll whose support from my days as an undergraduate was invaluable and to both he and Tom Ivey for agreeing to be my examiners. This work is dedicated to my parents Bernie and Noel without whose love, support and encouragement I would not have made it this far.

Abstract

We introduce the notion of flat surfaces of finite type in the 3-sphere, give the algebro-geometric description in terms of spectral curves and polynomial Killing fields, and show that finite type flat surfaces generated by curves on $\mathbb{S}^2$ with periodic curvature functions are dense in the space of all flat surfaces generated by curves on $\mathbb{S}^2$ with periodic curvature functions.

Chapter 1

Introduction

This thesis is devoted to the study of both flat surfaces in $\mathbb{S}^3$ and finite gap curves in $\mathbb{S}^2$ and, in particular, to combining the theory of both to produce flat surfaces of finite type in $\mathbb{S}^3$ generated by pairs of finite gap curves in $\mathbb{S}^2$.

1.1 Historical background

A surface in $\mathbb{S}^3$ is said to be flat if its Gaussian curvature K_{int} satisfies $K_{int} = 0$ at every point on the surface. Flat surfaces in $\mathbb{S}^3$ were already considered by Bianchi [4] in 1896 and by Spivak [29] in 1979 in terms of their asymptotic curves. The problem of classifying, in particular, those flat surfaces that are tori was first proposed by Yau [32] in 1974. Progress was made by Pinkall [26] in 1985 when he showed that given any curve on $\mathbb{S}^2$, its pre-image under the Hopf fibration is a flat surface called a Hopf cylinder and, in particular, if the curve on $\mathbb{S}^2$ is closed the resultant surface is compact and hence a (Hopf) torus. However this does not generate all flat surfaces or tori in $\mathbb{S}^3$ and the classification of all immersed flat tori was given independently, and by novel approaches by Kitagawa [23] in 1987, in terms of asymptotic lifts of pairs of curves on $\mathbb{S}^2$ that satisfy certain compatibility conditions (called admissibility conditions), Weiner [31] in 1989, who determines flat tori in terms of their Gauss map, and Bayard [1] who gives a spinor description of flat surfaces in $\mathbb{S}^3$. In 1972, Hasimoto [21] introduced the notion of the complex curvature of a curve given in terms of the curvature κ and torsion τ of the curve by $q = \kappa e^{i\int \tau(x)dx}$. It was shown that if the curve evolves according to the vortex filament equation (VFE), then the complex curvature satisfies the self-focusing nonlinear Schrödinger equation (NLS). In 1971 Zakharov and Shabat [33] showed that NLS is integrable and as a result has finite gap solutions. In 1997 Grinevich and Schmidt [19] studied periodic solutions of the VFE, while Calini and Ivey in a series of papers studied finite gap solutions of the VFE [5] and their isoperiodic deformations [6]. This work is concerned with the fusion of the theory of flat surfaces in $\mathbb{S}^3$ and that of finite gap curves.

1.2 Summary of contents

We follow Kitagawa's classification of immersed flat surfaces, wherein a flat surface is determined by two closed curves on $\mathbb{S}^2$ that satisfy some additional admissibility conditions. The two curves in turn give rise to two Hopf cylinders. We then choose the non fiber asymptotic curve from each Hopf cylinder, generating an asymptotic lift of each of the curves on $\mathbb{S}^2$ to curves in $\mathbb{S}^3$. The additional conditions on the two curves ensure that the group product of their asymptotic lifts yields an immersed flat surface. Kitagawa proves that all immersed flat surfaces arise in this way. In particular it is shown that flat tori are generated by pairs of admissible curves that are in addition periodic. Thus the description of immersed flat surfaces and, in particular, flat tori in $\mathbb{S}^3$ is reduced to the study of certain pairs of closed curves on $\mathbb{S}^2$. A summary of both the Spivak-Bianchi and Kitagawa approaches can also be found in [10]. Up to isometry a curve on $\mathbb{S}^2$ is determined by its geodesic curvature function. There is an interesting and well-studied infinite class of evolution equations on closed spherical curves called the mKdV hierarchy [15, 24]. The dynamics impose that the curve does not stretch during the deformation, that it stays on a sphere, but that the deformation does not depend on the radius of the sphere [8]. For a curve to be stationary under all flows it has to be a circle, since the curvature function is constant. For a curve to be stationary under all but the first flow means that it is critical for the bending energy of the curve, and this class includes all the wave-like and orbit-like elastic curves, as well as the generalized elastica [28]. If a curve does not change under the dynamics of a member of this hierarchy, it is called stationary under this evolution. A stationary curve is then automatically stationary under all the subsequent evolution equations in the mKdV hierarchy. The curves which are eventually stationary form an interesting class of curves on $\mathbb{S}^2$, and are called finite gap curves, or also finite type curves. A curve on $\mathbb{S}^2$ is finite gap if in the spectrum associated with its geodesic curvature function k_g, the difference between successive terms (called the gap length) is non zero for only finitely many terms. Equivalently a curve will be finite gap if all flows of sufficiently high order of the mKdV hierarchy restricted to this curve are linear combinations of lower order flows. We give an algebro-geometric description of these in terms of spectral curves, polynomial Killing fields, and in particular discuss the isoperiodic deformation [17, 19] in this setting. From a result due to Kappeler and Pöschel [22], we have that finite gap curves generated by periodic k_g are dense in the set of immersed curves with periodic geodesic curvature. From this we prove that immersed finite type flat surfaces generated by curves in $\mathbb{S}^2$ with periodic k_g are dense in the set of all flat immersed surfaces generated by curves with periodic geodesic curvature.

1.3 Summary of new results

We introduce the notion of flat surface of finite type in the 3-sphere as being the flat surface in $\mathbb{S}^3$ generated by the product of the asymptotic lifts of two finite gap curves in $\mathbb{S}^2$ following Kitagawa. We also give the algebro-geometric description of finite gap curves in terms of spectral curves and polynomial Killing fields along with a reconstruction formula following Sym [30]. We further show that finite type flat surfaces generated by curves on $\mathbb{S}^2$ with periodic k_g are dense in the space of all immersed flat surfaces in $\mathbb{S}^3$ generated by curves on $\mathbb{S}^2$ with periodic geodesic curvature. We also discuss the problem of attempting to extend this result to flat tori of finite type and outline why there is no effective way to ensure the closure of the finite gap curves using just the knowledge of the curvature function following a result of Nikolaevsky [25] which ensures that no finite list of integral equations given as a function of curvature, can generate all closed curves on $\mathbb{S}^2$. Lastly we discuss some potential avenues for further study by outlining some open questions in the area.

Chapter 2

The 3-Sphere

We begin with an introduction to $\mathbb{S}^3$ and a summary of some of its properties. We show that it is diffeomorphic to the set of 2×2 special unitary matrices with Lie algebra $\mathfrak{su}_2$. The theory of curves on $\mathbb{S}^3$ is developed and it is shown that a curve on $\mathbb{S}^3$ can be determined by two functions. Lastly we show the remarkable property that given a point on $\mathbb{S}^3$ and a great circle not containing p, there are exactly two great circles through p parallel to the given great circle.

2.1 Properties of $\mathbb{S}^3$

$\mathbb{S}^3$ or the unit 3-sphere is the set of unit vectors in $\mathbb{R}^4$, that is

$$\mathbb{S}^3 = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \mid x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Recall that two manifolds M and N are said to be diffeomorphic if there exists a differentiable bijection $f : M \to N$ whose inverse is also differentiable. $\mathbb{S}^3$ is a Lie group diffeomorphic to $SU_2(\mathbb{C})$, which is the group of 2x2 unitary matrices with determinant 1 given by

$$SU_2 = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

To see this, give $\mathbb{S}^3$ a group structure by identifying it with the unit quaternions

$$S_3 = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H} \mid a^2 + b^2 + c^2 + d^2 = 1\}$$

where the group operation is quaternion multiplication determined by $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ and inverses are given by conjugation. Moreover, since these group operations are smooth on $\mathbb{S}^3$ it is also a Lie group. We first show that $\mathbb{S}^3$ and SU_2 are isomorphic as Lie groups. There is a bijection $\mathbb{C}^2 \mapsto \mathbb{H}$ given by:
$(a+ib,\ c+id) \mapsto a+b\mathbf{i}+c\mathbf{j}-d\mathbf{k} = (a+b\mathbf{i})+\mathbf{j}(c+d\mathbf{i})$, that is, $(z_1, z_2) \mapsto z_1+\mathbf{j}z_2$.
Next observe that an element $y = w_1 + \mathbf{j}w_2 \in \mathbb{H}$ defines a mapping

$$\gamma : \mathbb{H} \to \mathbb{H} \text{ by } z_1 + \mathbf{j}z_2 \mapsto y \cdot (z_1 + \mathbf{j}z_2).$$

Noting that $z\mathbf{j} = \mathbf{j}\bar{z}$, we have that: $y \cdot (z_1 + \mathbf{j}z_2) = (w_1 + \mathbf{j}w_2) \cdot (z_1 + \mathbf{j}z_2) = w_1 z_1 + \mathbf{j}w_2\mathbf{j}z_2 + \mathbf{j}w_2 z_1 + w_1\mathbf{j}z_2 = (w_1 z_1 - \bar{w}_2 z_2) + \mathbf{j}(w_2 z_1 + \bar{w}_1 z_2)$. So ϕ is a linear map $\phi(y) : \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} w_1 z_1 - \bar{w}_2 z_2 \\ w_2 z_1 + \bar{w}_1 z_2 \end{pmatrix}$$

that is

$$\phi(y) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} w_1 & -\bar{w}_2 \\ w_2 & \bar{w}_1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

or

$$\phi(y) = \begin{pmatrix} w_1 & -\bar{w}_2 \\ w_2 & \bar{w}_1 \end{pmatrix}.$$

Thus there is a map $\phi : \mathbb{H} \to M_2(\mathbb{C})$, where $M_2(\mathbb{C})$ is the set of 2×2 complex valued matrices, given by $y \mapsto \phi(y)$.

Proposition 2.1.1. *SU_2 is diffeomorphic to $\mathbb{S}^3$*

Proof. Let $y \in \mathbb{S}^3$, then $\det(\phi(y)) = \bar{w}_1 w_1 + \bar{w}_2 w_2 = 1$ and $\phi(y)^*\phi(y) = 1$ where ϕ^* is the conjugate transpose of ϕ. So $\phi(y) \in SU_2$ and so restricting ϕ to $\mathbb{S}^3$ results in a map $\phi : \mathbb{S}^3 \to SU_2$. Conversely, let $A \in SU_2$ so that $\det(A) = 1$, $A^*A = I$ and A is of the form

$$A = \begin{pmatrix} w_1 & -\bar{w}_2 \\ w_2 & \bar{w}_1 \end{pmatrix} \text{ for some } w_1, w_2 \in \mathbb{C}.$$

Now choose $y = \phi^{-1}(A) = w_1 + \mathbf{j}w_2 \in \mathbb{H}$ then $\phi^{-1} : SU_2 \mapsto \mathbb{H}$ and $\phi(y) = A$. Moreover, since $\bar{y}y = \bar{w}_1 w_1 + \bar{w}_2 w_2 = 1$ it follows that $y \in \mathbb{S}^3$ and thus $\phi : \mathbb{S}^3 \to SU_2$ is a bijection. Finally, it is clear that both ϕ and ϕ^{-1} are smooth maps since their components are linear functions of the coordinates, and since

$\mathbb{S}^3$ and SU_2 are submanifolds, the restrictions of ϕ and ϕ^{-1} to these submanifolds are also smooth. Thus ϕ is a diffeomorphism and $\mathbb{S}^3$ is diffeomorphic to SU_2. □

Denote the Lie algebra of SU_2 by $\mathfrak{su}_2$. Let G be a Lie subgroup of $GL_n(\mathbb{C})$ and $g \in G$. Since G is a submanifold of $GL_n(\mathbb{C})$, the tangent space of G at g, T_gG, can be identified with

$$\{\gamma^{'}(0) \mid \gamma(0) = g \text{ and } \gamma : (-\epsilon, \epsilon) \mapsto G, \epsilon > 0, \text{ is smooth}\} \tag{2.1}$$

by mapping $\gamma^{'}(0)$ to the element of T_gG that acts on a smooth function f by $\frac{d}{dt}(f \circ \gamma)|_{t=0}$. Denote by $\mathfrak{g}$ the Lie algebra of G. Then since left multiplication is a diffeomorphism, (2.1) identifies T_gG with the set

$$g\mathfrak{g} = \{gX \mid X \in \mathfrak{g}\}.$$

Definition 2.1.2. Let G be a Lie subgroup of $GL_n(\mathbb{C})$ and $X \in \mathfrak{g}$.

1. Let $\bar{X}$ be the vector field on G defined by $\bar{X}_g = gX, g \in G$.
2. Let γ_X be the integral curve of $\bar{X}$ through $\mathbb{1}$, the identity, that is γ_X is the unique maximally defined smooth curve in G satisfying

$$\gamma_X(0) = \mathbb{1} \text{ and } \gamma_X(t) = \bar{X}_{\gamma_X}(t) = \gamma_X(t)X.$$

The following theorems will allow us to explicitly calculate the Lie algebra [27].

Theorem 2.1.3. *Let G be a Lie subgroup of $GL_n(\mathbb{C})$ and $X \in \mathfrak{g}$.*

1. *Then*
$$\gamma_X(t) = \exp(tX) = e^{tX} = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n.$$
2. *Moreover, γ_X is a homomorphism and complete, that is it is defined for all $t \in \mathbb{R}$ so that $e^{tX} \in G$ for all $t \in \mathbb{R}$.*

Proof. Note that $t \mapsto e^{tX}$ is a well-defined smooth homomorphism of $\mathbb{R}$ into $GL_n(\mathbb{C})$. Extend $\bar{X}$ to a vector field on $GL_n(\mathbb{C})$ by $\bar{X}_g = gX, g \in GL_n(\mathbb{C})$. Since $e^{0X} = \mathbb{1}$ and $\frac{d}{dt}e^{tX} = e^{tX}X$, $t \mapsto e^{tX}$ is the unique integral curve for $\bar{X}$ passing through $\mathbb{1}$ as a vector field on $GL_n(\mathbb{C})$. It is clearly complete. On the other hand, since G is a submanifold of $GL_n(\mathbb{C})$, γ_X may be viewed as a curve in $GL_n(\mathbb{C})$. It is still an integral curve for $\bar{X}$ passing through $\mathbb{1}$ as a vector field on $GL_n(\mathbb{C})$. By uniqueness, $\gamma_X = e^{tX}$ on the domain of γ_X. In particular there is an $\epsilon > 0$ so that $\gamma_X(t) = e^{tX}$ for $t \in (-\epsilon, \epsilon)$. Thus $e^{tX} \in G$ for $t \in (-\epsilon, \epsilon)$. But since $e^{ntX} = (e^{tX})^n$ for $n \in \mathbb{N}$, $e^{tX} \in G$ for all $t \in \mathbb{R}$, and this concludes the proof. □

Remark. Theorem 2.1.3 shows that the map $t \mapsto e^{tX}$ is a smooth map from $\mathbb{R}$ to G for $X \in \mathfrak{g}$

Theorem 2.1.4. *Let G be a Lie subgroup of* $GL_n(\mathbb{C})$ *with Lie algebra* $\mathfrak{g}$. *Then*

$$\mathfrak{g} = \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid e^{tX} \in G \textit{ for } t \in \mathbb{R}\}.$$

Proof. By Theorem 2.1.3, it follows that $\mathfrak{g}$ is contained in

$$\{X \in \mathfrak{gl}_n(\mathbb{C}) \mid e^{tX} \in G \text{ for } t \in \mathbb{R}\}.$$

Conversely if $e^{tX} \in G$ for $t \in \mathbb{R}$ for all $X \in \mathfrak{gl}_n(\mathbb{C})$, applying $\frac{d}{dt}|_{t=0}$ to e^{tX} and using the definition it follows that $X \in \mathfrak{g}$ concluding the theorem. □

Proposition 2.1.5. $\mathfrak{su}_2 = \{U \in \mathfrak{gl}_2(\mathbb{C}) \mid U^* = -U, \operatorname{tr}(U) = 0\}$ *where* $\mathfrak{gl}_2(\mathbb{C})$ *is the Lie algebra of* $GL_2(\mathbb{C}) = M_2(\mathbb{C})$.

Proof. Firstly, it is shown that the Lie algebra of

$$U(2) = \{U \in M_2(\mathbb{C}) \mid U^*U = UU^* = I\}$$

is

$$\mathfrak{u}_2 = \{U \in \mathfrak{gl}_2(\mathbb{C}) \mid U^* = -U\}.$$

Suppose that X is in the Lie algebra of $U(2)$. Then, by the previous theorem, $1 = (e^{tX})(e^{tX})^* = e^{tX}(e^{tX})^*, t \in \mathbb{R}$. Applying $\frac{d}{dt}|_{t=0}$ implies that $0 = X + X^*$ and hence $X^* = -X$. On the other hand, if $X^* = -X$, then $e^{tX}(e^{tX})^* = 1$, so that X is in the Lie algebra of $U(2)$. Then to calculate the Lie algebra of SU_2 we add the determinant condition. If X is in the Lie algebra of SU_2 then: $1 = \det(e^{tX}) = e^{t(\operatorname{tr}(X))}$, $t \in \mathbb{R}$. Applying $\frac{d}{dt}|_{t=0}$ implies that $0 = \operatorname{tr}(X)$. Conversely, if $\operatorname{tr}(X) = 0$, then $\det(e^{tX}) = e^{t(\operatorname{tr}(X))} = 1$, so that X is in the Lie algebra. Thus the Lie algebra of SU_2 is:

$$\mathfrak{su}_2 = \{X \in \mathfrak{gl}_2(\mathbb{C}) \mid X^* = -X, \operatorname{tr}(X) = 0\}.$$

□

Theorem 2.1.6. *Great circles are geodesics on $\mathbb{S}^3$.*

Proof. A curve $\gamma : \mathbb{R} \to \mathbb{S}^3$ is a geodesic if and only if γ satisfies

$$\gamma''(t) = \langle \gamma''(t), \gamma \rangle \gamma. \quad [7]$$

Let C be a great circle in $\mathbb{S}^3$ and without loss of generality assume that $C(t) = \{\cos(t), \sin(t), 0, 0\}$. Then a calculation shows that $\langle C''(t), C \rangle = -1$ and that $C''(t) = -C(t)$. Thus C is a geodesic on $\mathbb{S}^3$. $\square$

2.2 Curves in $\mathbb{S}^3$

We develop the theory of curves in $\mathbb{S}^3$ by generating a Frenet-Serret series of equations in $\mathbb{R}^4$ consisting of three curvatures and showing that for curves in $\mathbb{S}^3$ the third of these is determined by the other two. Let $\gamma : \mathbb{R} \to \mathbb{R}^4$ be an arclength parameterized curve with arclength parameter s. Let $t_1 = \gamma'$ denote the unit tangent vector of γ. Then since $\langle t_1, t_1 \rangle = 1$ we have

$$0 = \frac{d}{ds} \langle t_1(s), t_1(s) \rangle = 2 \left\langle t_1(s), \frac{dt_1(s)}{ds} \right\rangle.$$

We then define the first curvature function κ_1 by

$$\kappa_1(s) = \left| \frac{dt_1(s)}{ds} \right| = |\gamma''(s)|.$$

If $\kappa_1(s) \neq 0$ for all s we set $t_2(s) = \kappa_1(s)^{-1} \dfrac{dt_1(s)}{ds}$. Then t_2 is a unit vector field along γ and $\langle t_1(s), t_2(s) \rangle = 0$ for all s. Moreover

1. $\dfrac{dt_1(s)}{ds} = \kappa_1(s) t_2(s)$.
2. $\langle t_2(s), t_2(s) \rangle = 1$.

(2) implies that $\langle t_2(s), \frac{dt_2(s)}{ds} \rangle = 0$ and $\langle t_1(s), t_2(s) \rangle = 0$ implies that

$$0 = \left\langle \frac{dt_1(s)}{ds}, t_2(s) \right\rangle + \left\langle t_1(s), \frac{dt_2(s)}{ds} \right\rangle$$

that is

$$0 = \kappa_1(s) + \left\langle t_1(s), \frac{dt_2(s)}{ds} \right\rangle$$

and thus

$$\frac{dt_2(s)}{ds} = -\kappa_1(s)t_1(s) + n(s)$$

where $n(s)$ satisfies $\langle t_1, n\rangle = 0 = \langle t_2, n\rangle$. We then define the second curvature function by

$$\kappa_2(s) = \left|\frac{dt_2(s)}{ds} + \kappa_1(s)t_1(s)\right| = |n(s)|.$$

If $\kappa_2(s) \neq 0$ for all s we set

$$t_3(s) = \kappa_2(s)^{-1}\left(\frac{dt_2(s)}{ds} + \kappa_1(s)t_1(s)\right)$$

so that t_3 is a unit length vector field along γ that satisfies $\langle t_1, t_3\rangle = \langle t_2, t_3\rangle = 0$. Moreover

$$\frac{dt_2(s)}{ds} = -\kappa_1(s)t_1(s) + \kappa_2(s)t_3(s).$$

Then $\langle t_3, t_3\rangle = 1$ implies that $\left\langle t_3, \frac{dt_3}{ds}\right\rangle = 0$, and $\langle t_1, t_3\rangle = 0$ implies that

$$0 = \left\langle \frac{dt_1}{ds}, t_3\right\rangle + \left\langle t_1, \frac{dt_3}{ds}\right\rangle$$

and thus $0 = \left\langle t_1, \frac{dt_3}{ds}\right\rangle$. We also have that since $\langle t_2, t_3\rangle = 0$

$$0 = \left\langle \frac{dt_2}{ds}, t_3\right\rangle + \left\langle t_2, \frac{dt_3}{ds}\right\rangle$$

thus

$$0 = \kappa_2 + \left\langle t_2, \frac{dt_3}{ds}\right\rangle.$$

Hence $\frac{dt_3}{ds} = -\kappa_2(s)t_2(s) + b(s)$ where $\langle b(s), t_1(s)\rangle = \langle b(s), t_2(s)\rangle = \langle b(s), t_3(s)\rangle = 0$. The third curvature is then given by

$$\kappa_3\left|\frac{dt_3}{ds} + \kappa_2 t_2\right| = |b(s)|.$$

Similarly if $\kappa_3(s) \neq 0$ for all s we can define

$$t_4(s) = \kappa_3(s)^{-1}\left(\frac{dt_3}{ds} + \kappa_2 t_2\right)$$

and since $t_4(s)$ is a unit length vector field that has 0 inner product with t_1, t_2 and t_3 we can derive that

$$\frac{dt_3}{ds} = -\kappa_2(s)t_2(s) + \kappa_3(s)t_4(s).$$

We thus have the Frenet-Serret style matrix for $\mathbb{R}^4$ [29] given by

$$\begin{pmatrix} t_1'(s) \\ t_2'(s) \\ t_3'(s) \\ t_4'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(s) & 0 & 0 \\ -\kappa_1(s) & 0 & \kappa_2(s) & 0 \\ 0 & -\kappa_2(s) & 0 & \kappa_3(s) \\ 0 & 0 & -\kappa_3(s) & 0 \end{pmatrix} \begin{pmatrix} t_1(s) \\ t_2(s) \\ t_3(s) \\ t_4(s) \end{pmatrix}$$

Theorem 2.2.1. *Let γ be a curve on a sphere in $\mathbb{R}^4$, then γ is determined by any two of the three curvature functions $\kappa_1,\ \kappa_2,\ \kappa_3$.*

Proof. Let γ be a curve on the sphere in $\mathbb{R}^4$ centered at $a \in \mathbb{R}^4$ with radius r. Then $\langle \gamma(s) - a, \gamma(s) - a \rangle = r$. Differentiating yields

1. $2\langle t_1(s), \gamma(s) - a \rangle = 0$.
2. $\langle t_1'(s), \gamma(s) - a \rangle = -1$.

Now $\kappa_1(s) = \|t_1'(s)\| \neq 0$ and so $t_2(s)$ is well defined by $t_1'(s) = \kappa_1(s)t_2(s)$. Hence

$$\langle t_2(s), \gamma(s) - a \rangle = -\frac{1}{\kappa_1(s)}.$$

Differentiating again we obtain

$$\begin{aligned} 0 &= \kappa_1'(s)\langle t_2(s), \gamma(s) - a \rangle + \kappa_1(s)\langle t_2'(s), \gamma(s) - a \rangle + \kappa_1(s)\langle t_2(s), t_1(s) \rangle \\ &= \kappa_1'(s) \left\langle \frac{1}{\kappa_1(s)} t_1'(s), \gamma(s) - a \right\rangle + \kappa_1(s)\langle -\kappa_1(s)t_1(s) + \kappa_2(s)t_3(s), \gamma(s) - a \rangle \\ &= \frac{\kappa_1'(s)}{\kappa_1(s)}\langle t_1'(s), \gamma(s) - a \rangle + \kappa_1(s)\kappa_2(s)\langle t_3(s), \gamma(s) - a \rangle \\ &= -\frac{\kappa_1'(s)}{\kappa_1'(s)} + \kappa_1(s)\kappa_2(s)\langle t_3(s), \gamma(s) - a \rangle. \end{aligned} \tag{2.2}$$

Differentiating once more yields

$$\begin{aligned}
0 =& \frac{d}{ds}\left(-\frac{\kappa_1'(s)}{\kappa_1(s)}\right) + \kappa_1'(s)\kappa_2(s)\langle t_3(s), \gamma(s) - a\rangle + \kappa_1(s)\kappa_2'(s)\langle t_3(s), \gamma(s) - a\rangle \\
&+ \kappa_1(s)\kappa_2(s)\langle -\kappa_2(s)t_2(s) + \kappa_3(s)t_4(s), \gamma(s) - a\rangle \\
=& \frac{d}{ds}\left(-\frac{\kappa_1'(s)}{\kappa_1(s)}\right) + \langle t_3(s), \gamma(s) - a\rangle(\kappa_1'(s)\kappa_2(s) + \kappa_1(s)\kappa_2'(s)) + \kappa_2^2(s) \\
&+ \kappa_1(s)\kappa_2(s)\kappa_3(s)\langle t_4(s), \gamma(s) - a\rangle.
\end{aligned}$$

By (2.2), $\langle t_3(s), \gamma(s) - a\rangle = \dfrac{\kappa_1'(s)}{\kappa_1^2(s)\kappa_2(s)}$. Thus

$$\begin{aligned}
0 =& \frac{d}{ds}\left(-\frac{\kappa_1'(s)}{\kappa_1(s)}\right) + \frac{\kappa_1'(s)}{\kappa_1^2(s)\kappa_2(s)}(\kappa_1'(s)\kappa_2(s) + \kappa_1(s)\kappa_2'(s)) + \kappa_2^2(s) \\
&+ \kappa_1(s)\kappa_2(s)\kappa_3(s)\langle t_4(s), \gamma(s) - a\rangle \\
=& \frac{d}{ds}\left(-\frac{\kappa_1'(s)}{\kappa_1(s)}\right) + \frac{\kappa_1'^2(s)}{\kappa_1^2} + \frac{\kappa_1'(s)\kappa_2'(s)}{\kappa_1(s)\kappa_2(s)} + \kappa_2^2(s) \\
&+ \kappa_1(s)\kappa_2(s)\kappa_3(s)\langle t_4(s), \gamma(s) - a\rangle.
\end{aligned} \tag{2.3}$$

Now $\{t_1(s), t_2(s), t_3(s), t_4(s)\}$ is an orthonormal basis of $\mathbb{R}^4$ and hence

$$\begin{aligned}
r^2 =& \langle \gamma(s) - a, \gamma(s) - a\rangle^2 \\
=& \langle \gamma(s) - a, t_1(s)\rangle^2 + \langle \gamma(s) - a, t_2(s)\rangle^2 + \langle \gamma(s) - a, t_3(s)\rangle^2 \\
&+ \langle \gamma(s) - a, t_4(s)\rangle^2 \\
=& 0 + \left(\frac{1}{\kappa_1^2(s)}\right) + \left(\frac{\kappa_1'(s)}{\kappa_1^2(s)\kappa_2(s)}\right)^2 + \langle t_4(s), \gamma(s) - a\rangle^2.
\end{aligned}$$

By (2.3)

$$\langle t_4(s), \gamma(s) - a\rangle = \frac{1}{p}\left(-\frac{d}{ds}\left(-\frac{\kappa_1'(s)}{\kappa_1(s)}\right) - \frac{\kappa_1'^2(s)}{\kappa_1^2(s)} - \frac{\kappa_1'(s)\kappa_2'(s)}{\kappa_1(s)\kappa_2(s)} - \kappa_2^2(s)\right)$$

where $p = \kappa_1(s)\kappa_2(s)\kappa_3(s)$. Then a calculation shows that

$$\begin{aligned}
\kappa_3(s) = \frac{1}{q}\Big(&\kappa_1^2(s)\kappa_2(s)\kappa_1''(s) - \kappa_1(s)\kappa_1'^2(s)\kappa_2(s) \\
&- \kappa_1(s)\kappa_1'^2(s)\kappa_2(s) - \kappa_1^2(s)\kappa_1'(s)\kappa_2'(s) - \kappa_1^3(s)\kappa_2^3(s)\Big)
\end{aligned}$$

where $q = \kappa_1^4(s)\kappa_2^2(s)r^2 - \kappa_1^2(s)\kappa_2^2(s) - \kappa_1'^2(s)$. Thus curves on $\mathbb{S}^3$ are determined by their first and second curvatures. □

2.3 Clifford Parallelism

[2] This section is devoted to describing the unique property of circle parallelism in $\mathbb{S}^3$, in particular we will show:

Theorem. *For each great circle C in $\mathbb{S}^3$, $\alpha \in (0, \frac{\pi}{2})$ and point m on the great circle C_α of distance α from C there exists exactly two (Clifford) parallel circles to C.*

We begin by defining the distance between two great circles in $\mathbb{S}^3$.

Definition 2.3.1. 1. Define a metric d on S^3 by

$$d(u, v) = \cos^{-1}(\langle u, v\rangle)$$

where $\langle u, v\rangle$ is the dot product.

2. Let $m \in \mathbb{S}^3$, C a great circle of $\mathbb{S}^3$. Then the distance between m and C is given by

$$d(m, C) = \inf\{d(m, v) \mid v \in C\}.$$

3. For any two great circles C_1 and C_2 of S^3 define

$$d(C_1, C_2) = \inf\{d(u, C_2) \mid u \in C_1\}.$$

4. Say that any two great circles C_1 and $C_2 \subset S^3$ are Clifford parallel if $d(u, C_2) = d(v, C_2)$ for all $u, v \in C_1$.

5. For every great circle C of S^3 and every $\alpha \in (0, \pi)$ we set $C_\alpha = \{n \in S^3 \mid d(n, C) = \alpha\}$. Moreover, since C contains antipodal points it follows that $C_{\pi-\alpha} = C_\alpha$ and so it is possible to restrict to $\alpha \in (0, \frac{\pi}{2})$. Denote by P_C the plane in $\mathbb{R}^4$ such that $C = P_C \cap S^3$ and by $C^\perp$ the great circle in S^3 such that $P_{C^\perp} = (P_C)^\perp$.

Now introduce an orthonormal basis (x, y, z, t) on S^3 whose first two basis vectors are in C and the other two in $C^\perp$ and complexifying we set $u = x + iy, v = z + it$.

Proposition 2.3.2. *For $m \in S^3, m = (x_1, y_1, z_1, t_1) = (u_1, v_1)$, $d(m, C) = \alpha$ is equivalent to $x_1^2 + y_1^2 = |u_1| = \cos^2 \alpha$.*

Proof. Since C is spanned by the first two basis vectors and $C \subset S^3$, C will have the form $(\omega, 0) \in \mathbb{C}^2$ where ω is a unit vector. Then

$d((u_1, v_1), (\omega, 0)) = \cos^{-1}(\langle (u_1, v_1), (\omega, 0)\rangle)$ will be minimal when $\langle (u_1, v_1), (\omega, 0)\rangle$ is maximal and this occurs when $\omega = \frac{u_1}{|u_1|}$ in which case
$\cos\alpha = \langle (u_1, v_1), (\frac{u_1}{|u_1|}, 0)\rangle = \frac{|u_1|^2}{|u_1|} = |u_1| = \sqrt{x_1^2 + y_1^2}$. Thus $\cos^2\alpha = x_1^2 + y_1^2$. □

Furthermore, note that since $(u_, v_1) \in S^3$, $|u_1|^2 + |v_1|^2 = 1$, and this implies that $|v_1|^2 = z_1^2 + t_1^2 = \sin^2\alpha$. Next we turn to quadratic forms to give us a characterization of great circles in $\mathbb{S}^3$ as the intersection of $\mathbb{S}^3$ with a cone generated by a neutral quadratic q. This will allow us to determine properties of the totally singular planes in $(\mathbb{R}^4, q)$ which we can use to generate a second Clifford parallel to a given great circle.

Definition 2.3.3. Let E and E' be Euclidean spaces of dimension $2p$, then:

1. Given two quadratic forms q on E and $q^{'}$ on $E^{'}$, q and $q^{'}$ are called equivalent if there exists an isomorphism $f : E \to E^{'}$ such that $q = f^* q^{'}$.
2. A quadratic form is called *neutral* if it is equivalent to $q = 2\sum_{i=1}^{p} x_i y_i$, where x_i and y_i are the coordinates on E.

Lemma 2.3.4. *Let $C_\alpha = \{m \in \mathbb{S}^3 \mid d(m, C) = \alpha\}$. Then $C_\alpha = S^3 \cap Q_\alpha$ where Q_α is the cone in $\mathbb{R}^4$ given by $\sin^2\alpha(x^2 + y^2) - \cos^2\alpha(z^2 + t^2) = 0$.*

Proof. From proposition 2.3.2 it follows that for a given great circle C: $d(n, C) = \alpha \Leftrightarrow x^2 + y^2 = \cos^2\alpha$, $z^2 + t^2 = \sin^2\alpha$. So if $m \in C_\alpha$, $m = (x, y, z, t)$, then m satisfies $\sin^2\alpha(x^2 + y^2) - \cos^2\alpha(z^2 + t^2) = 0$. Hence $C_\alpha \subset S^3 \cap Q_\alpha$. Conversely, if $m \in S^3 \cap Q_\alpha$, then

$$\sin^2\alpha(x^2 + y^2) = \cos^2\alpha(z^2 + t^2) \text{ and } x^2 + y^2 + z^2 + t^2 = 1.$$

This implies that $x^2 + y^2 = \cos^2\alpha$ which implies that $m \in C_\alpha$ and so $S^3 \cap Q_\alpha \subset C_\alpha$ which implies that $C_\alpha = S^3 \cap Q_\alpha$. □

Note. Moreover note that $q = \sin^2\alpha(x^2 + y^2) - \cos^2\alpha(z^2 + t^2)$ is a neutral quadratic.

Definition 2.3.5.

1. Associated to a quadratic q is a map ϕ (its Polar Form) given by $\phi(x, y) = \frac{1}{2}(q(x + y) - q(x) - q(y))$. Also associated to ϕ is a morphism $\psi \in L(E, E^*)$ given by setting $\psi(x)(y) = \phi(x, y)$.
2. The radical of q is $Ker(\psi) = \{x \in E \mid \phi(x, y) = 0 \text{ for any } y \in E\}$.
3. q is said to be non-degenerate if $rad(q) = 0$. Other wise it is said to be degenerate.

4. $F \subset E$ is said to be non-singular (respectively singular) if $q|_F$ is non-degenerate (respectively degenerate). F is said to be completely singular if $q|_F \equiv 0$ that is if $rad(F) = F$.

5. A line (or plane) is said to be isotropic if it is contained in $q^{-1}(0)$.

6. Let q be a non-degenerate quadratic form. Then the group of q is $O(q) = \{f \in GL(E) \mid f^*(q) = q\}$, $O^+(q) = \{f \in O(q) \mid \det(f) = +1\}$.

Lemma 2.3.6. *Let Γ be the set of completely singular planes of $(\mathbb{R}^4, q)$ where q is a neutral quadratic and let Π and $\sum$ be the two distinct orbits of Γ under the action of $O^+(q)$ [3]. Then*

(i) Any isotropic line D of $(\mathbb{R}^4, q)$ is contained in a unique plane $P \in \prod$ and in a unique $S \in \sum$.

(ii) For any $P \in \prod$ and any $S \in \sum$, $P \cap S$ has dimension 1 (and so must be an isotropic line).

(iii) For any planes $P, P' \in \prod$ either $P \cap P' = 0$ or $P = P'$ and similarly for planes in $\sum$.

Proof. Let $\mathbb{R}^4 = \{(x, y, z, t) \mid x, y, z, t \in \mathbb{R}\}$, then by definition $q = 2xy + 2zt$ since q is neutral. Then there are two sets of isotropic planes, one of the form $\lambda x = -\mu z, \mu y = \lambda t$ and another of the form $\lambda x = -\mu t, \mu y = \lambda z$ with $(\lambda, \mu) \neq (0, 0)$. Properties (i)-(iii) then follow. $\square$

Thus we are now ready to prove

Theorem 2.3.7. *For each great circle $C \subset S^3$, $\alpha \in (0, \frac{\pi}{2})$ and $m \in C_\alpha$, there exists exactly two Clifford parallels to C.*

Proof. (a) There exists at least one Clifford Parallel. Consider the Hopf map $h : S^3 \to S^2$ given by $h(z_1, z_2) = \frac{z_1}{z_2}$. Now the fibers of h are great circles and moreover S^3 is a union of these fibers and any two fibers of h are Clifford parallel since the action of S^1 on S^3 given by $\lambda(z_1, z_2) = (\lambda z_1, \lambda z_2)$, $\lambda \in S^1$ is isometric. So choosing $m \in C_\alpha$ there is a fiber C_1 of h with $m \in C_1$ and C_1 parallel to C.

(b) There exists two Clifford Parallels. The great circle C_1 in (a) is the result of an intersection of a plane P passing through the origin intersected with S^3 and moreover $C_1 \subset C_\alpha$. Thus $q|_p \equiv 0$ since q is 0 at the origin and on a circle in the plane that is P is completely isotropic. Now since $C_1 \subset P$ and $m \in C_1$ implies that $-m \in C_1 \subset P$. Let L be the line in P joining m to

$-m$ so that $L \subset P$ and then L is completely isotropic. Then by lemma 2.3.6 (part (i)) there exists a plane S with $q|_S \equiv 0$ and $P \cap S = L$. Now $0 \in S$ since $0 \in L$ implies that $S \cap S^3$ is a great circle C_2 with $m \in C_2$ and since $P \cap S = L$, $C_1 \neq C_2$. Moreover C_2 is Clifford parallel to C since q is neutral that is $q|_{C_2} \equiv 0$ which implies that $C_2 \subset Q_\alpha \cap S^3 = C_\alpha$.

(c) There exist exactly two Clifford parallels. Note that there can only be two Clifford parallels as suppose there existed some circle C_3 Clifford parallel to C and without loss of generality, suppose $C_3 \subset P_2 \in \prod$. Then $m \in C_3 \cap C_1$ but $P_2 \cap P = \emptyset$ or $P = P_2$, that is $C_3 = C_1$. Similarly if $P_2 \in \sum$.

□

Remark. In particular, this shows that Clifford parallelism is not an equivalence relation since C_1 is not Clifford parallel to C_2 as
$m \in C_1 \cap C_2$ which implies that $d(m, C_i) = 0$ that is the distance between them does depend on the point.

Chapter 3

Surface Theory

We devote this chapter to an introduction to surface theory with the focus on first developing the theory of flat surfaces in $\mathbb{S}^3$ which we describe wholly in terms of their asymptotic curves. We begin by recalling the essential concepts.

3.1 Preliminaries

Definition 3.1.1. (i) Let Σ be a surfaces in $\mathbb{S}^3$, then Σ is said to be a *regular surface* if for each $p \in \Sigma$ there exists a neighbourhood V of p in $\mathbb{R}^4$ and a surjective map $x : U \to V \cap \Sigma$, U an open subset of $\mathbb{R}^2$, such that

(a) x is differentiable, injective and x^{-1} is differentiable.

(b) The Jacobian of x has rank two at every point in U.

x is then called a surface patch.

(ii) Let (M, g) be a Riemannian manifold and $C^\infty(M, TM)$ be the space of vector fields on M. Then the Levi-Civita connection on M is the bilinear map $D : C^\infty(M, TM) \times C^\infty(M, TM) \mapsto C^\infty(M, TM)$, $(X, Y) \mapsto D_X Y$, such that for all smooth functions $f \in C^\infty(M, \mathbb{R})$ D satisfies:

(a) $D_{fX}Y = fD_X Y$.

(b) $D_X(fY) = df(X)Y + fD_X Y$.

(c) $Dg = 0$ where g is the Riemannian metric on M.

(d) $D_X Y - D_Y X = XY - YX$.

(iii) Let M be a regular surface, $p \in M$ and $X, Y \in T_pM$. Then the first fundamental form of M denoted by I is the inner product of the tangent vectors, that is $I(X, Y) = \langle X, Y \rangle$. It is given explicitly by

$$I = Edu^2 + 2Fdudv + Gdv^2$$

where if $x(u, v)$ is a surface patch, $E = \langle x_u, x_u \rangle$, $F = \langle x_u, x_v \rangle$ and $G = \langle x_v, x_v \rangle$.

(iv) Let Σ be oriented and let N be its unit normal along Σ compatible with the orientation. Then the second fundamental form of Σ is $II(X, Y) = \langle -D_X N, Y \rangle$ where X and Y are smooth vector fields in Σ and D is the Levi-Civita connection of S^3. Furthermore if $x(u, v)$ is a surface patch and N the unit normal to Σ, then II is given as the quadratic form

$$ldu^2 + 2mdudv + ndv^2$$

where $l = \langle x_{uu}, N \rangle$, $m = \langle x_{uv}, N \rangle$ and $n = \langle x_{vv}, N \rangle$.

(v) For $p \in \Sigma$, the principal curvatures at p, denoted by κ_1 and κ_2 are the eigenvalues of the matrix

$$\begin{pmatrix} II(X_1, X_1) & II(X_1, X_2) \\ II(X_2, X_1) & II(X_2, X_2) \end{pmatrix}$$

where $\{X_1, X_2\}$ is an orthonormal basis of tangent vectors at p.

(vi) The mean curvature of Σ at p is $H(p) = \dfrac{\kappa_1 + \kappa_2}{2}$.

(vii) The Gauss map is the map $\eta : \Sigma \to \mathbb{S}^3$ that maps p to the vector normal to Σ at p.

(viii) Let X and Y be smooth manifolds, $f : X \to Y$ a smooth map. Then the pushforward of f is the map $f_* : TX \to TY$ given by: if $v \in T_pX$ we can represent v as the tangent to a curve $c : I \to X$ with $c(0) = p$. Then $f_*(v) = (f \circ c)'(0)$. Let $\phi : Y \to \mathbb{R}$ be a smooth function, then the pullback of ϕ by f is the map $f^*\phi : \Sigma \to \mathbb{R}$ given by $(f^*\phi)(x) = \phi(f(x))$.

(ix) Let X, Y and Z be tangent vector fields on a Riemannian manifold M. Then the Riemannian curvature tensor is the tensor

$$R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$$

where $[X,Y] = \frac{1}{2}(XY - YX)$ and D is the Levi-Civita connection on M.

(x) Let M be a Riemannian manifold, u and v two linearly independent tangent vectors to M at $p \in M$. Then the sectional curvature of M at p is given by

$$K_{sectional}(p) = \frac{\langle R(u,v)v, u\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}.$$

3.2 Flat Surfaces in $\mathbb{S}^3$

Note. For a surface $\Sigma \subset \mathbb{S}^3$ there is an important distinction between intrinsic and extrinsic curvature, the surface Σ has an induced Riemannian metric and thus has an intrinsic curvature:

$$K_{int}(p) = \frac{\langle R(X_p, Y_p)Y_p, X_p\rangle}{\langle X_p, X_p\rangle\langle Y_p, Y_p\rangle - \langle X_p, Y_p\rangle^2}$$

for orthonormal $X_P, Y_p \in T_p\Sigma$. It also has an extrinsic Gaussian curvature

$$K_{ext}(p) = k_1(p).k_2(p)$$

given by the product of the principal curvatures at p.

For surfaces in $\mathbb{R}^3$ these two concepts coincide and for a 3-manifold of constant sectional curvature they are related by

Proposition 3.2.1. *Let $(N, \langle\, ,\, \rangle)$ be a 3-dimensional manifold of constant sectional curvature K_0 and $\Sigma \subset N$ a surface. Then the intrinsic curvature of Σ and the extrinsic curvature are related-by*

$$K_{int} = K_{ext} + K_0.$$

Proof. Denote by D' and D the covariant derivative in N and Σ respectively, with D determined by the metric $i^*\langle\, ,\, \rangle$ where i is the inclusion map. For each $p \in \Sigma$ we have an orthogonal splitting $T_pN = T_p\Sigma \oplus T_p\Sigma^\perp$ into the tangent and normal spaces respectively and their associated projection maps
$\perp : T_pN \mapsto T_p\Sigma^\perp$ and $\top : T_pN \mapsto T_p\Sigma$. Given this terminology, for vector fields X and Y tangent along Σ we write

$$D'_{X_p}Y = \top(D'_{X_p}Y) + \perp(D'_{X_p}Y).$$

Moreover

$$\top(D'_{X_p}Y) = D_{X_p}Y \text{ and } \perp(D'_{X_p}Y) = s(X_p, Y_p) = II(X_p, Y_p) \cdot v(p)$$

are symmetric in X_p and Y_p. where $v(p)$ is the vector normal to Σ at p. Thus from this follows the Gauss Formula:

$$D'_{X_p}Y = D_{X_p}Y + s(X_p, Y_p).$$

Moreover, denote by R and R' the curvature tensors of N and Σ respectively, we derive from this Gauss' Equation [29]:

$$\langle R'(X,Y)Z, W\rangle = \langle R(X,Y)Z, W\rangle + \langle s(X,Z), s(Y,W)\rangle - \langle s(Y,Z), s(X,W)\rangle$$

where X,Y,Z,W are tangent vector fields. Then recalling the definition of intrinsic curvature

$$K_{int}(p) = \frac{\langle R(X_p,Y_p)Y_p, X_p\rangle}{\langle X_p, X_p\rangle\langle Y_p, Y_p\rangle - \langle X_p, Y_p\rangle^2}$$

and sectional curvature

$$\frac{\langle R(X,Y)Y, X\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}$$

the result follows. □

Definition 3.2.2. We define a surface $\Sigma \subset \mathbb{S}^3$ to be flat if its intrinsic curvature satisfies $K_{int} = 0$ (which is equivalent to $K_{ext} = -1$).

3.3 Asymptotic Curves on Surfaces

To understand flat surfaces in $\mathbb{S}^3$ and in particular the classification of flat tori, we turn to a study of the generating curves of these surfaces.

Proposition 3.3.1. *Let X_1, X_2 be linearly independent vector fields in a neighbourhood of a point p in a 2-dimensional manifold Σ. Then there is an embedding $f : U \to \Sigma$, $U \subset \mathbb{R}^2$ open and $p \in f(U)$, whose i^{th} parameter lines lie along the integral curves of X_i.*

Proof. It can be assumed without loss of generality that $p = 0 \in \mathbb{R}^2$ and that $X_i(0) = (e_i)_0$ where $\{e_i\}$ is the basis for the tangent space. Every point q in a sufficiently small neighbourhood of 0 is on a unique integral curve of X_1

through a point $(0, x_2(q))$. Similarly, q is on a unique integral curve of X_2 through a point $(x_1(q), 0)$. The map $q \mapsto (x_1(q), x_2(q))$ is C^∞ with Jacobian equal to the identity at 0. Its inverse in a sufficiently small neighbourhood of 0 is then the required diffeomorphism. □

Definition 3.3.2. Let Σ be a surface, then:

(i) The normal curvature k_n of a unit-speed curve $c : \mathbb{R} \to \Sigma$ is defined to be $k_n(s) = -\langle d\nu(c'(s)), c'(s)\rangle = II(c'(s), c'(s))$ where ν is the Gauss map and II is the second fundamental form.

(ii) A curve $c : \mathbb{R} \to \Sigma$ is said to be asymptotic if its normal curvature satisfies $\kappa_n(t) = 0$ for all t.

We will show that all (complete) flat surfaces in $\mathbb{S}^3$ are determined by their asymptotic curves. To prove the existence of asymptotic curves we first need:
Lemma 3.3.3. *(Euler's Theorem) Let Σ be a surface, $c : I \to \Sigma$ a curve, then the normal curvature of c is given by*

$$\kappa_n(s) = k_1(c(s)) \cos^2\theta + k_2(c(s)) \sin^2\theta$$

where k_1 and k_2 are the principal curvatures and θ is the oriented angle from the principal direction X_1 (corresponding to k_1) to $c'(s)$.

Proof.

$$\begin{aligned}
\kappa_n(s) &= \langle d\nu(c'(s)), c'(s)\rangle \\
&= \langle k_1(\cos\theta)X_1 + k_2(\sin\theta)X_2, (\cos\theta)X_1 + (\sin\theta)X_2\rangle \\
&= k_1(\cos^2\theta) + k_2(\sin^2\theta)
\end{aligned}$$

where X_2 is the principal direction corresponding to k_2. □

Proposition 3.3.4. *Let Σ be a surface, $p \in \Sigma$ with $K_{ext}(p) < 0$. Then there are exactly two asymptotic directions at p.*

Proof. By Euler's Theorem, it follows that the normal curvature of a curve γ passing through p is given by

$$\kappa_n(p) = k_1(p) \cos^2\theta + k_2(p) \sin^2\theta$$

where k_1 and k_2 are the principal curvatures and θ is the angle between γ' and the principal direction t_1 corresponding to k_1. Solving

$$k_1 \cos^2\theta + k_2 \sin^2\theta = 0$$

yields

$$\theta = \cos^{-1}\left(\pm\sqrt{-\frac{k_2}{k_1 - k_2}}\right)$$

and hence there exists two asymptotic directions at p. □

Note. As a result of these two propositions we have that in a small region of a point p on a surface Σ where $K_{ext}(p) < 0$, we may choose two linearly independent asymptotic unit vectors X_p and Y_p at each point p. Then $p \mapsto X_p$ and $p \mapsto Y_p$ will be C^∞ vector fields whose integral curves are asymptotic. Thus there exists an embedding $f : U \to \Sigma$ with $p \in U$ whose parameter curves (that is the curves $f(u, v_1)$ and $f(u_1, v)$ where u_1 and v_1 are constants) are asymptotic curves.

Theorem 3.3.5. *Let Σ be an immersed 2-dimensional submanifold of a 3-dimensional manifold N, with constant extrinsic curvature $K_{ext} < 0$. Then for every point $p \in \Sigma$ there is a diffeomorphism*

$$\begin{aligned} &g : (-\epsilon,\, \epsilon) \times (-\epsilon,\, \epsilon) \to \Sigma \\ &g(0,\, 0) = p \end{aligned}$$

whose parameter curves are asymptotic curves parameterized by arclength.

Proof. From the previous note, it follows that for every $p \in \Sigma$ there exists a diffeomorphism g satisfying the first two requirements above whose parameter curves are asymptotic curves. By a suitable reparametrization we can clearly arrange that the two parameter curves through $p = g(0,\, 0)$ are parameterized by arclength. Thus in the first fundamental form

$$E(s, 0) = 1, G(0, t) = 1.$$

Moreover we show that all parameter curves are parameterized by arclength. Let

$$\begin{aligned} I &= Edu^2 + 2Fdudv + Gdv^2 \\ II &= ldu^2 + 2mdudv + ndv^2 \end{aligned}$$

be the first and second fundamental forms of Σ. Then the Codazzi-Mainardi equations are

$$
\begin{aligned}
l_v - m_u &= l\Gamma^1_{12} + m(\Gamma^2_{12} - \Gamma^1_{11}) - n\Gamma^2_{11} \\
m_v - n_u &= l\Gamma^1_{22} + m(\Gamma^2_{22} - \Gamma^1_{12}) - n\Gamma^2_{12}
\end{aligned}
$$

where the Christoffel symbols are given by

$$
\begin{aligned}
\Gamma^1_{11} &= \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)} \\
\Gamma^2_{11} &= \frac{2EF_u - 2EE_v - FE_u}{2(EG - F^2)} \\
\Gamma^1_{12} &= \frac{GE_v - FG_u}{2(EG - F^2)} \\
\Gamma^2_{12} &= \frac{EG_u - FE_v}{2(EG - F^2)} \\
\Gamma^1_{22} &= \frac{2GF_v - GG_u - FG_u}{2(EG - F^2)} \\
\Gamma^1_{11} &= \frac{EG_v - 2FF_v + FE_u}{2(EG - F^2)}.
\end{aligned}
$$

Then since the parameter lines are asymptotic, we have $l = n = 0$. It is possible to rewrite the Codazzi-Mainardi equations as

$$
(m^2)_u = 2\left(\frac{\frac{1}{2}(EG - F^2) + FE_v - EG_u}{EG - F^2}\right) m^2 \tag{3.1}
$$

$$
(m^2)_v = 2\left(\frac{\frac{1}{2}(EG - F^2) + FG_u - EG_u}{EG - F^2}\right) m^2. \tag{3.2}
$$

Furthermore

$$
K_{ext} = \frac{ln - m^2}{EG - F^2} = -\frac{m^2}{EG - F^2}
$$

so that

$$
m^2 = (-K_{ext})(EG - F^2).
$$

Substituting this into (3.1), and using the fact that K_{ext} is a negative constant, we have

$$
(-K_{ext})(EG - F^2)_u = 2(-K_{ext})(\tfrac{1}{2}(EG - F^2)_u + FE_v - EG_u).
$$

Hence $EG_u - FE_v = 0$. Similarly, substituting into (3.2) yields
$-FG_u + GE_v = 0$. Then since $EG - F^2 \neq 0$, these equations imply that $E_v = 0$

and $G_u = 0$ and thus combined with (1) it follows that E and G must equal one everywhere. $\square$

Note. We call such a diffeomorphism a Tchebychev net. Moreover given such a Tchebychev net the surface is determined in terms of its first and second fundamental forms by:

Corollary 3.3.6. *Let Σ be a surface with constant extrinsic curvature $K_{ext} < 0$ and let $(I,\, II)$ be its first and second fundamental forms. Then there exist local parameters $(u,\, v)$ and a smooth function $\omega(u,\, v) \in (0, \pi)$ such that*

$$I = du^2 + 2\cos\omega du dv + dv^2$$
$$II = 2\sqrt{-K_{ext}}\sin\omega du dv.$$

Proof. From theorem 3.3.5 there exist local coordinates (given by the diffeomorphism g), such that in the first fundamental form $E = G = 1$ and in the second fundamental form $L = N = 0$. Moreover, since $K_{ext} = -\dfrac{m^2}{1-F^2}$ is a negative constant it follows that $F^2 + \left(\dfrac{m}{\sqrt{-K_{ext}}}\right)^2 = 1$. Hence there exists a smooth function ω such that

$$I = du^2 + 2\cos\omega du dv + dv^2$$
$$II = 2\sqrt{-K_{ext}}\sin\omega du dv$$

with $0 < \omega < \pi$ since $1 - F^2 > 0$. $\square$

Next we show that the angle between the tangents to the asymptotic curves at a point is related to the intrinsic curvature and splits into the sum of two single variable functions

Proposition 3.3.7. *Let Σ be a 2-dimensional Riemannian manifold with $g : (a, b) \times (c, d) \to \Sigma$ a Tchebychev net. Define $\omega(u_0, v_0)$ to be the unique angle between*

$$\left.\frac{dg(s, t_0)}{ds}\right|_{s=s_0} \text{ and } \left.\frac{dg(s_0, t)}{dt}\right|_{t=t_0}.$$

Then

$$\frac{\partial^2\omega}{\partial s \partial t} = (-K_{int})\sin\omega.$$

Proof. Since g is a Tchebychev net, it follows that $E = G = 1$ and $F = \cos\omega$.

Let $W = \sqrt{EG - F^2} = \sin\omega$, Then

$$\begin{aligned} K_{int} &= \frac{1}{2W}\left[\frac{\partial}{\partial t}\left(\frac{F_u}{W}\right) + \frac{\partial}{\partial s}\left(\frac{F_v}{W}\right)\right] \\ &= \frac{1}{2\sin\omega}\left[\frac{\partial}{\partial t}\left(-\frac{\partial\omega}{\partial s}\right) + \frac{\partial}{\partial s}\left(-\frac{\partial\omega}{\partial t}\right)\right] \\ &= \left(\frac{\partial^2\omega}{\partial s\partial t}\right)\left(-\frac{1}{\sin\omega}\right). \end{aligned}$$

□

Note. In particular on a flat surface Σ, with local coordinates (u, v), ω satisfies $\omega_{uv} = 0$ and hence by d'Alembert's formula $\omega(u, v) = \omega_1(u) + \omega_2(v)$ where ω_1 and ω_2 are smooth functions.

Furthermore, we can characterize the torsion (second curvature) of all asymptotic curves.

Definition 3.3.8. The generalized cross product in $\mathbb{R}^4$ of three vectors a, b and c is given by

$$a \times b \times c = \sum_{i=1}^{i=4} \det(e_i abc) \cdot e_i$$

where $\{e_i\}$ are the standard unit basis vectors in $\mathbb{R}^4$.

Theorem 3.3.9. *Let Σ be a surface and $\phi : \Sigma \to \mathbb{S}^3$ an immersion with negative constant extrinsic curvature K_{ext}. Then the asymptotic curves of ϕ have constant torsion τ with $\tau^2 = -K_{ext}$ at points where the curvature is non-zero. Moreover, two asymptotic curves through a point have torsions of opposite signs if their curvature is non-zero at that point.*

Proof. By corollary 3.3.6 there exist local coordinates (u, v) such that

$$\begin{aligned} I &= du^2 + 2\cos\omega du dv + dv^2 \\ II &= 2\sqrt{-K_{ext}}\sin\omega du dv. \end{aligned}$$

With this parametrization, the coordinate curves are the asymptotic curves of the immersion. Considering the asymptotic curve $\alpha(u) = \phi(u, v_0)$, which is parameterized by arclength, then

$$\langle D_{\alpha'(u)}\alpha'(u),\, N(u, v_0)\rangle = 0$$

where N is the normal to the immersion. Hence if $\kappa(u_0) \neq 0$, then up to a factor of ± 1, $N(u_0, v_0)$ is the binormal vector to the curve at $u = u_0$. The normal vector of $\alpha(u)$ is then $\alpha \times N(u,\, v_0) \times \alpha'(u)$ at the points where $\kappa \neq 0$. Write the normal to the curve as $a(u)\frac{\partial}{\partial u} + b(u)\frac{\partial}{\partial v}$. Then

$$\begin{aligned}
\tau(u) &= -\langle D_{\alpha'(u)} N(u,\, v_0),\, \alpha \times N(u,\, v_0) \times \alpha'(u)\rangle \\
&= -\langle D_{\alpha'(u)} N(u,\, v_0),\, a(u)\tfrac{\partial}{\partial u} + b(u)\tfrac{\partial}{\partial v}\rangle \\
&= b(u)\langle N(u,\, v_0),\, D_{\frac{\partial}{\partial u}} \tfrac{\partial}{\partial v}\rangle \\
&= b(u)\sqrt{-K_{ext}}\,\sin\omega(u,\, v_0).
\end{aligned}$$

We calculate $b(u)$ by

$$\begin{aligned}
0 &= \langle N(u,\, v_0) \times \alpha'(u), \tfrac{\partial}{\partial u}\rangle = a(u) + b(u)\cos\omega \\
\sin\omega &= \langle N(u,\, v_0) \times \alpha'(u),\, \tfrac{\partial}{\partial v}\rangle = a(u)\cos\omega + b(u).
\end{aligned}$$

Hence $b(u) = \dfrac{1}{\sin\omega}$ and thus $\tau = -\sqrt{-K_{ext}}$. $\square$

Chapter 4

Flat Tori in $\mathbb{S}^3$

To begin the discussion of flat tori we introduce the first non-homogeneous class of flat tori, generated by the Hopf map. We show that the inverse image of any curve on $\mathbb{S}^2$ under the Hopf map is a flat surface and if the curve is closed it is a flat torus. We then develop the Bianchi-Spivak method [4], [29] of generating flat surfaces in $\mathbb{S}^3$ using asymptotic curves in $\mathbb{S}^3$ whose torsion satisfies $\tau = \pm 1$, before specializing this approach to classifying flat tori following Kitagawa [23] by using pairs of curves on $\mathbb{S}^2$ that satisfy certain admissibility conditions.

4.1 Hopf Tori

Identify $\mathbb{R}^4$ with the quaternions by $(x_1, x_2, x_3, x_4) \mapsto x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4$ and $\mathbb{S}^3$ as the space of unit quaternions. Then $\mathbb{S}^2 = \mathbb{S}^3 \cap \{x_1 = 0\}$ is the space of purely imaginary unit quaternions.

Definition 4.1.1. The Hopf map is the map $h : \mathbb{S}^3 \to \mathbb{S}^2$ given by $h(x) = x\mathbf{i}\bar{x}$, where for $x = x_1 + ix_2 + jx_3 + kx_4$, $\bar{x} = x_1 - ix_2 - jx_3 - kx_4$.

If $p = \mathbf{i}p_2 + \mathbf{j}p_3 + \mathbf{k}p_4$ is a point in $\mathbb{S}^2$, then

$$h^{-1}(p) = \left\{ x \in \mathbb{S}^3 \,\middle|\, \begin{array}{l} 0 = x_1(p_2 - 1) - p_4x_3 + p_3x_4, \\ 0 = x_2(p_2 - 1) + p_3x_3 + p_4x_4 \end{array} \right\}, \text{ if } p_2 \neq 1$$

$$h^{-1}(\mathbf{i}) = \left\{ x \in \mathbb{S}^3 \mid x_1 = 0 = x_2 \right\}$$

which are the intersections of $\mathbb{S}^3$ with planes through the origin and hence are great circles.

Note. The Hopf map may be written in terms of coordinates in $\mathbb{R}^4$ by

$$h(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2 - x_3^2 - x_4^2, 2(x_1x_4 + x_2x_3), 2(x_2x_4 - x_1x_3)).$$

Our first example of a flat torus is the Clifford torus given by

$$\{(\cos\theta, \sin\theta, \cos\phi, \sin\phi) \mid 0 \le \theta < 2\pi, 0 \le \phi < 2\pi\}$$

which is a product torus of two great circles in $\mathbb{S}^3$. It is a flat torus as a result of the following lemma:

Lemma 4.1.2. *If two families of geodesics intersect at a constant angle everywhere on a surface $\Sigma \in \mathbb{S}^3$, then Σ is flat.*

Proof. Let X (respectively Y) be the vector field of unit tangent vectors to the curves of the first (respectively second) family. Then X is parallel along the integral curves of X. Since the angle between X and Y is constant and since we are on a surface, Y must be parallel along the integral curves of X. Similarly for X along Y. Thus

$$0 = D_X X = D_Y Y = D_X Y = D_Y X = [X, Y]$$

and hence

$$K_{int} = R(X, Y)Y = D_X(D_Y Y) - D_Y(D_X Y) - D_{[X,Y]}Y = 0.$$

□

Using this lemma we can now generate a large class of flat surfaces using the Hopf map.

Theorem 4.1.3. *Let c be an immersed curve in $\mathbb{S}^2$. Then $h^{-1}(c)$ is a flat surface in $\mathbb{S}^3$.*

Proof. Firstly, suppose that c is a circle of radius R, with $0 < R < 1$. Then up to isometry, it may be assumed that c lies in a plane parallel to the xy-plane. Then, letting (p_2, p_3, p_4) be coordinates on $\mathbb{S}^2$, it follows that $p_2^2 + p_3^2 = R^2$ and

$p_4 = \sqrt{1-R^2}$. Hence

$$h^{-1}(c) = \left\{ x \in \mathbb{S}^3 \middle| \begin{array}{l} 0 = x_1(p_2-1) - \sqrt{1-R^2}x_3 + p_3x_4 \\ 0 = x_2(p_2-1) + p_3x_3 + \sqrt{1-R^2}x_4 \\ p_2^2 + p_3^2 = R^2 \end{array} \right\}.$$

This is the product of two circles in perpendicular planes and thus is the product torus $S^1(R) \times S^1(\sqrt{1-R^2})$. By the preceding lemma is thus a flat surface. Now let c be any immersed curve. For any $q \in \mathbb{S}^2$, consider the osculating circle $C \subset \mathbb{S}^2$ of c at q (that is C is the circle with contact of order two at q that is C is tangent to c at q and the curvature of C as a curve in $\mathbb{S}^2$ is the same as the curvature of c at q). Then $h^{-1}(c)$ and $h^{-1}(C)$ agree up to second order at $h^{-1}(c(q))$ and since $h^{-1}(C)$ is a flat torus with $K_{ext} = -1$, $h^{-1}(c)$ must also have $K_{ext} = -1$ and hence is flat. □

Note. Such surfaces are called Hopf cylinders and if c is closed then $h^{-1}(c)$ is compact and is called a Hopf torus. In particular we note that since if c is a great circle in $\mathbb{S}^2$, $h^{-1}(c)$ is a product torus made up of families of great circles and in particular we have that the fibers of h are asymptotic curves.

4.2 Bianchi-Spivak Method

We now turn to a development of the theory of flat surfaces in terms of their asymptotic curves. We will show that all complete flat surfaces in $\mathbb{S}^3$ can be written as a product of asymptotic curves. This is known as the Bianchi-Spivak method [29]. Let $\phi : \Sigma \to \mathbb{S}^3$ be a flat immersion of a surface Σ into $\mathbb{S}^3$ with unit normal N. Then $K_{int} = 0$ implies that $K_{ext} = -1$ and thus there exists a Tchebychev net on Σ. We first show that all asymptotic curves on a flat surface are the same up to isometry, so that determining two of them determines the surface up to isometry.

Proposition 4.2.1. *All the asymptotic curves $u_1 \to \phi(u_1, u_2)$ are congruent to each other. Similarly all the asymptotic curves $u_2 \to \phi(u_1, u_2)$ are all congruent to each other.*

Proof. Write $e_i = \frac{\partial}{\partial u_i}$, $n_i = N \times e_i$ where N is the normal to the immersion.

Frenet-Serret then implies that

$$\begin{aligned} D_{e_i} e_i &= \kappa_i n_i \\ D_{e_i} n_i &= \kappa_i e_i + \tau_i N \\ D_{e_i} N &= -\tau_i n_i. \end{aligned}$$

Note that, as in the proof of Theorem 3.3.9, N is the binormal to the curves, n_i their normals, κ_i their curvature and τ_i their torsion. Moreover, $\tau_1 = 1, \tau_2 = -1$. Then $\kappa_i = \langle D_{e_i} e_i, n_i \rangle$. Let $D_{e_1} e_1 = a e_1 + b e_2$ for suitable a and b. Then

$$\begin{aligned} \kappa_1 &= \langle a e_1 + b e_2, N \times e_1 \rangle \\ &= b \langle e_2, N \times e_1 \rangle. \end{aligned}$$

Then, since the curves are asymptotic, it follows that n_1 lies in the tangent plane, hence $\langle e_2, n_1 \rangle = \|e_2\| \, \|n_1\| \cos(\omega - \frac{\pi}{2}) = \sin \omega$, where ω is the angle between the tangents to the asymptotic curves. Hence $\kappa_1 = b \sin \omega$ and b can be calculated by taking the inner product of $D_{e_1} e_1$ with e_i

$$\begin{aligned} a + b\cos\omega &= \langle D_{e_1} e_1, e_1 \rangle = 0 \\ a\cos\omega + b &= \langle D_{e_1} e_1, e_2 \rangle = e_1 \langle e_1, e_2 \rangle = -\omega_u \sin\omega. \end{aligned}$$

Then, by the note following Definition 4.1.1, $\omega(u_1, u_2) = \omega_1(u_1) + \omega_2(u_2)$ and it follows that $\kappa_1(u_1) = \omega_1'(u_1)$. Similarly $\kappa_2(u_2) = \omega_2'(u_2)$. Hence any asymptotic curves $u_1 \mapsto \phi(u_1, u_2)$ has curvature and torsion which do not depend on u_2 and similarly for $u_2 \mapsto \phi(u_1, u_2)$ as required. $\square$

Moreover a flat surface is completely determined by its asymptotic curves by:
Theorem 4.2.2. *The flat immersion $\phi(u_1, u_2)$ can be recovered in terms of the asymptotic curves $\phi(u_1, 0)$ and $\phi(0, u_2)$ as the quaternionic product*

$$\Psi(u_1, u_2) = \phi(u_1, 0)\phi(0, 0)^{-1}\phi(0, u_2).$$

Proof. First we compute

$$\begin{aligned}
\Psi_{u_1} &= \phi'(u_1,0)\phi(0,0)^{-1}\phi(0,u_2)\\
\Psi_{u_1u_2} &= \phi'(u_1,0)\phi(0,0)^{-1}\phi'(0,u_2)\\
\Psi_{u_1u_1} &= \phi''(u_1,0)\phi(0,0)^{-1}\phi(0,u_2)\\
\Psi_{u_2} &= \phi(u_1,0)\phi(0,0)^{-1}\phi'(0,u_2)\\
\Psi_{u_2u_2} &= \phi(u_1,0)\phi(0,0)^{-1}\phi''(0,u_2).
\end{aligned}$$

Hence in the first fundamental form $E = G = 1$ and since the parameter curves are asymptotic it follows that in the second fundamental form that $L = N = 0$. Then

$$\begin{aligned}
F &= \langle \phi'(u_1,0)\phi(0,0)^{-1}\phi(0,u_2), \phi(u_1,0)\phi(0,0)^{-1}\phi'(0,u_2)\rangle\\
M &= \langle \phi(u_1,0)\phi(0,0)^{-1}\phi'(0,u_2), \bar{N}\rangle
\end{aligned}$$

where $\bar{N}$ is the unit normal to the immersion. Then since $\cos\omega = \langle \phi'(u_1,0), \phi'(0,u_2)\rangle$ and all the asymptotic curves are congruent, $\phi'(u_1,0)$ is left invariant, similarly $\phi'(0,u_2)$ is right invariant and hence $F = \cos\omega$. To calculate M, recall that $\det(S) = K_{ext}$ where S is the shape operator given by

$$S = (EG - F^2)^{-1}\begin{pmatrix} LG - MF & MG - NF \\ ME - LF & NE - MF \end{pmatrix}.$$

Hence $-M^2\sin^2\omega = -\sin^4\omega$ and thus $M = \sin\omega$. It then follows that the first and second fundamental forms agree with the Tschebyscheff coordinates in corollary 3.3.6 and moreover the initial conditions for ϕ and Ψ are the same and hence, by the Fundamental Theorem of Surfaces: $\phi(u_1,u_2) = \Psi(u_1,u_2)$. □

Note. Up to isometry it may be assumed that $\phi(0,0) = 1$. Furthermore letting $\xi_0 = N(0,0)$ be a vector in $\mathbb{S}^2$ normal to the immersion at $(0,0)$ then the normal is given by $\phi(u_1,0)\xi_0\phi(0,u_2)$.

The condition given in Theorem 3.3.9, that the second curvature of the asymptotic curves is equal to $\sqrt{-K_{ext}}$ in the flat case reduces to $\tau = \pm 1$ and this can be characterized by:

Proposition 4.2.3. *Let $\xi_0 \in \mathbb{S}^2 \subset \mathbb{S}^3$ where $\mathbb{S}^2$ is the set of unit purely imaginary quaternions. Let $\gamma(t)$ be a curve in $\mathbb{S}^3$ parameterized by arclength. Then if* *(i)* $\langle \gamma', \gamma\xi_0 \rangle = 0$, *then* $\gamma(t)$ *has torsion equal to 1 at points where its curvature is non-zero.*

(ii) $\langle \gamma', \xi_0\gamma \rangle = 0$, *then* $\gamma(t)$ *has torsion equal to -1 at points where its curvature is non-zero.*

Proof. Assume $\langle \gamma', \gamma\xi_0 \rangle = 0$. Then

$$\begin{aligned} 0 &= \langle D_{\gamma'}\gamma', \gamma\xi_0 \rangle + \langle \gamma', \gamma'\xi_0 \rangle \\ &= \langle D_{\gamma'}\gamma', \gamma\xi_0 \rangle \end{aligned}$$

as $\langle \gamma', \gamma'\xi_0 \rangle = 0$ since $\xi_0 \in \mathbb{S}^2$. Similarly $\langle \gamma, \gamma\xi_0 \rangle = 0$. Thus $\gamma\xi_0$ is orthogonal to γ, γ' and $D_{\gamma'}\gamma'$. Thus if $D_{\gamma'}\gamma' \neq 0$ that is if $\kappa \neq 0$, then $\gamma\xi_0$ is the binormal to the curve and hence the normal to γ is then $-\gamma'\xi_0$. Torsion is then given by

$$\tau = -\langle D_{\gamma'}\gamma\xi_0, -\gamma'\xi_0 \rangle = \langle \gamma'\xi_0, \gamma'\xi_0 \rangle = 1.$$

This proves (i). The proof of (ii) is similar. □

Note. If $\langle \gamma', \gamma\xi_0 \rangle = 0$ then $0 = \overline{\langle \gamma', \gamma\xi_0 \rangle} = -\langle \overline{\gamma}', \xi_0\overline{\gamma} \rangle$ since $\overline{\xi_0} = -\xi_0$, that is if γ has torsion 1, then $\overline{\gamma}$ has torsion -1 at points where $\kappa \neq 0$.

Thus flat surfaces in $\mathbb{S}^3$ are generated by pairs of curves that satisfy

Theorem 4.2.4. *Let* $\xi \in \mathbb{S}^2 \subset \mathbb{S}^3$. *Let* α *and* β *be arclength parameterized curves in* $\mathbb{S}^3$ *satisfying*

(i) $\langle \alpha', \alpha\xi_0 \rangle = 0$, $\langle \beta', \beta\xi_0 \rangle = 0$,

(ii) $\alpha(0) = \mathbb{1} = \beta(0)$,

(iii) $\overline{\alpha(u_1)}\alpha'(u_1) \neq \pm\overline{\beta(u_2)}\beta'(u_2)$ *for all* u_1, u_2.

Then $\phi(u_1, u_2) = \alpha(u_1)\overline{\beta(u_2)}$ *is a flat immersion with unit normal* $N(u_1, u_2) = \alpha(u_1)\xi_0\overline{\beta(u_2)}$.

Proof. Note that (iii) is equivalent to saying that $\phi_{u_1} \neq \pm\phi_{u_2}$, that is, ϕ is an immersion if and only if (iii) holds. (i) implies that α has torsion 1 and β has torsion -1 at points where their curvature is non-zero. By the proof of proposition 4.2.3 it follows that the normal to α is $-\alpha'\xi_0$ and the normal to β is $-\beta'\xi_0$. It then follows that $\langle D_{\alpha'}\alpha', -\alpha'\xi_0 \rangle = 0$ since $D_{\alpha'}\langle \alpha', \alpha'\xi_0 \rangle = 0$ as $\xi_0 \in \mathbb{S}^3$ and similarly for β. Thus α and β have normal curvature 0 and are thus asymptotic curves. Then by 3.3.9 it follows that $\phi(u_1, u_2) = \alpha\overline{\beta}$ is a flat immersion with normal $N = \alpha\xi_0\beta$. □

4.3 The Kitagawa Representation

Lastly we provide a summary of the classification of flat surfaces and in particular flat tori following Kitagawa [23]. The approach will be to generate lifts of pairs of curves in $\mathbb{S}^2$ to the non-fibre (that is non-great circle) asymptotic curves on $h^{-1}(c)$. We begin by developing the frame theory of $\mathbb{S}^3$. Let

$$e_i = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

be a basis of $\mathfrak{su}_2$. Define a left invariant vector field on SU_2 by

$$E_i(a) = \frac{d}{dt}(a \cdot \exp(te_i))|_{t=0}.$$

Then $\{E_1, E_2, E_3\}$ is a frame field (that is the set $\{E_i\}$ is a set of three orthonormal vector fields) on SU_2 and it is possible to define a vector product on each tangent space of $\mathbb{S}^3$ by $E_i \times E_j = \frac{1}{2}[E_i, E_j]$. Denote by D the Riemannian connection on $\mathbb{S}^3$ induced by the metric $\langle E_i, E_j \rangle = \delta_i^j$. Then $D_{E_i}E_j = \frac{1}{2}[E_i, E_j]$. We recall the definition of a Tchebychev net from Theorem 3.3.5.

Definition 4.3.1. An immersion $F : \mathbb{R}^2 \to \mathbb{S}^3$ is said to be a flat asymptotic Tchebychev immersion if F induces a flat metric on $\mathbb{R}^2$ and satisfies

$$\langle F_i, F_i \rangle = 1, \ \ \langle D_{F_i}F_i, \xi \rangle = 0$$

where

$$F_i = \frac{\partial F}{\partial t_i}, \ \ \xi = \frac{F_1 \times F_2}{\|F_1 \times F_2\|}.$$

Let $\mathbb{S}^2 = \{x \in \mathfrak{su}_2 \mid \|x\| = 1\}$. Recall the Hopf fibration $h : \mathbb{S}^3 \to \mathbb{S}^2$ given by $h(a) = Ad(a)e_3 = ae_3a^{-1}$. For each $x \in \mathbb{S}^2$, identify $T_x\mathbb{S}^2$ with a linear subspace $x^\perp$ of $\mathfrak{su}_2$ given by

$$x^\perp = \{y \in \mathfrak{su}_2 \mid \langle x, y \rangle = 0\}.$$

Let $\mathbb{S}^1$ be the closed subgroup of $\mathbb{S}^3$ given by

$$\mathbb{S}^1 = \{a \in \mathbb{S}^3 \mid Ad(a)e_3 = e_3\}.$$

The horizontal subspace H_a of $T_a\mathbb{S}^3$ is given by

$$H_a = \{v \in T_a\mathbb{S}^3 \mid \langle v, E_3 \rangle = 0\}.$$

The correspondence $a \mapsto H_a$ defines a connection on the principal $\mathbb{S}^1$-bundle.
Remark. Note that having defined such a connection, every vector in $x^\perp$ has a unique lift to a horizontal vector in H_y where y is a point in the fiber over x. Thus given a curve $\gamma : \mathbb{R} \to \mathbb{S}^2$ and considering the Hopf map $h : \mathbb{S}^3 \to \mathbb{S}^2$ as a principal fiber bundle, we can define a horizontal lift of γ to $\mathbb{S}^3$ to be the integral curve resulting from the lift of the tangent vectors to γ to horizontal vectors in $\mathbb{S}^3$.

Definition 4.3.2. Let M and N be manifolds, $f : M \to N$ a smooth map, X a vector field on M and Y a vector field on N. Then X and Y are said to be f related if $Y \circ f^* = f^* \circ X$.

We show that the covariant derivative of two horizontal lifts $\hat{X}$ and $\hat{Y}$ are related to the covariant derivatives of X and Y by h. Firstly we require the following lemmas.

Lemma 4.3.3. *If $v \in H_a$, then $\|h_*(v)\| = 2\|v\|$ where h is the Hopf fibration.*

Proof. Let $\gamma : \mathbb{R} \to \mathbb{S}^2$ be a curve, h the Hopf fibration, then if we identify $\mathbb{S}^2$ with the unit sphere in the space spanned by 1, $\mathbf{j}$ and $\mathbf{k}$ $h : \mathbb{S}^3 \to \mathbb{S}^2$ can be written as $h(q) = \tilde{q}q$ where if $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$, $\tilde{q} = q_0 - \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$. Without loss of generality, assume that γ is unit speed and let η be the unit speed horizontal lift of γ to $\mathbb{S}^3$. Note that if A, $B \in \mathbb{H}$, $\langle A, B\rangle_{\mathbb{H}} = Re(A\overline{B}) = Re(\overline{A}B) = \frac{A\overline{B}+B\overline{A}}{2}$. Thus A is orthogonal to B if and only if $A\overline{B} = -B\overline{A}$ or $\overline{A}B = -\overline{B}A$. Since η is unit speed it follows that η' is orthogonal to η. Hence $\langle \eta', \eta\rangle = 0$. Thus

$$\eta'\overline{\eta} = -\eta\overline{\eta'} \text{ and } \overline{\eta'}\eta = -\overline{\eta}\eta'.$$

Set $f(x, y) = e^{ix}\eta(y)$. Then f_x is orthogonal to f_y, that is $ie^{ix}\eta(y)$ is orthogonal to $e^{ix}\eta'(y)$ and thus $i\eta(y)$ is orthogonal to $\eta'(y)$. Thus $\langle i\eta, \eta'\rangle = 0$. Hence

$$i\eta\overline{\eta'} = \eta'\overline{\eta}i \text{ and } -\overline{\eta}i\eta' = -\overline{\eta'}i\eta.$$

Now $\gamma = h \circ \eta$ and $p(q) = \tilde{q}q = -i\overline{q}iq$. Hence

$$\begin{aligned}\gamma' &= -i\overline{\eta'}i\eta - i\overline{\eta}i\eta' \\ &= -i\overline{\eta}i\eta' - i\overline{\eta}i\eta' \\ &= -2i\overline{\eta}i\eta' \\ &= 2\gamma.\end{aligned}$$

Thus $\|\gamma'\| = 2\|\gamma\| = 2$. Hence if $v \in H_a$, then $\|h_*(v)\| = 2\|v\|$. □

Definition 4.3.4. Let M and B be Riemannian manifolds. Then

1. A Riemannian submersion is a mapping $f : M \to B$ of M onto B satisfying
   (a) $Df_p : T_pM \to T_pB$ is a surjective linear map.
   (b) $Df : ker(Df)^\perp \to TB$ is an isometry.
2. A vector field X on M is said to be basic if
   (a) X is horizontal, that is $\langle X(p), f^{-1}(b)\rangle = 0$ where $f : M \to B$ is a Riemannian submersion and $b \in B$ such that $p \in f^{-1}(b)$.
   (b) $f_*(X_p) \in T_{f(p)}B$ for all $p \in M$.

Lemma 4.3.5. *Let X and Y be basic vector fields on M and $f : M \to B$ a Riemannian submersion. Then*

1. $\langle X, Y\rangle = \langle X_*, Y_*\rangle \circ f$.
2. *Let H denote the projection of TM onto the subspace of horizontal vectors associated to f. Then $H([X, Y])$ is the basic vector field corresponding to $[X_*, Y_*]$.*
3. *$H(D_XY)$ is the basic vector field corresponding to $D^*_{X_*}(Y^*)$.*

Proof. (1) follows from the definition of a submersion. (2) follows from

$$f_*[X, Y] = [X_*, Y_*].$$

(3) For a basic vector field Z we have

$$\begin{aligned}2\langle D_XY, Z\rangle =& X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle \\ &+ \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle.\end{aligned}$$

But

$$X\langle Y, Z\rangle = X\{\langle Y_*, Z_*\rangle \circ f\} = X_*\langle Y_*, Z_*\rangle \circ f$$

and similarly for the other terms on the right hand side above, thus the right hand side is equal to

$$2\langle D^*_{X_*}Y_*, Z_*\rangle \circ f.$$

Denote by ∇ the Riemannian connection on $\mathbb{S}^2$ with respect to the standard metric. Then It follows that $D_X Y$ is f-related to $D^*_{X_*} Y_*$ where D^* is the Riemannian connection on B, and hence $H(D_X Y)$ is basic and corresponds to the covariant derivative. □

Thus as a result of these lemmas we have:

Proposition 4.3.6. *Let X and Y be vector fields on $\mathbb{S}^2$ and let $\hat{X}$ and $\hat{Y}$ be horizontal lifts of X and Y respectively. Then $D_{\hat{X}}\hat{Y}$ and $\nabla_X Y$ are h-related.*

Proof. Consider a new metric on $\mathbb{S}^3$ homothetic to the standard metric which has constant Gaussian curvature (given by the Riemannian curvature tensor) equal to 4. Then it follows from lemma 4.3.3 that $h : \mathbb{S}^3 \to \mathbb{S}^2$ is a Riemannian submersion with respect to the new metric. Since the new metric and the standard metric induce the same Riemannian connection D, the result follows from lemma 4.3.5. □

Define a $(1, 1)$-tensor field on $\mathbb{S}^2$ by $J(v) = \frac{1}{2}[x, v]$ for $v \in T_x\mathbb{S}^2$ and a $(1, 1)$-tensor field on $\mathbb{S}^3$ by $\tilde{J}(E_i) = E_3 \times E_i$. Then by proposition 4.3.6 we conclude

Lemma 4.3.7. $h_* \circ \tilde{J} = J \circ h_*$.

Let $\gamma : \mathbb{R} \to \mathbb{S}^2$ be a regular curve. Then $h^{-1}(\gamma)$ is a Hopf cylinder and we define:

Definition 4.3.8. A curve $c : \mathbb{R} \to \mathbb{S}^3$ is said to be an asymptotic lift of γ if $h \circ c = \gamma$ and c is an asymptotic curve on $h^{-1}(\gamma)$.

We will use a horizontal lift of a curve to generate an asymptotic lift of γ to $\mathbb{S}^3$. Let $p : \mathbb{R} \to \mathbb{S}^3$ be a horizontal lift of γ and let θ be a real valued function on $\mathbb{R}$. Define $c : \mathbb{R} \to \mathbb{S}^3$ by

$$c(t) = p(t)exp(\theta(t)e_3).$$

Recalling that the geodesic curvature k_g of a curve γ on a surface is given by

$$k_g = \frac{\langle \gamma''(t), (n \times \gamma'(t)) \rangle}{\|\gamma'(t)\|^3}$$

or equivalently

$$k_g = \frac{\langle D_{\gamma'}\gamma', J(\gamma') \rangle}{\|\gamma'(t)\|^3}.$$

A necessary and sufficient condition for c to be an asymptotic lift of γ is given by:

Lemma 4.3.9. *The curve c is an asymptotic lift of γ if and only if $\theta' = \dfrac{k_g\|\gamma'\|}{2}$.*

Proof. Let ξ be a vector field along c given by

$$\xi = -\frac{\tilde{J}(c')}{\|\tilde{J}(c')\|}.$$

c is an asymptotic lift of γ if and only if $\langle D_{c'}c', \xi\rangle = 0$. Thus it is enough to show that

$$\langle D_{c'}c', \xi\rangle = \theta'\|\gamma'\| - \frac{1}{2}k_g\|\gamma'\|^2. \tag{4.1}$$

Now

$$c' = (R_{exp(\theta e_3)})_* h' + \theta' E_3(c). \tag{4.2}$$

Let $a_i = \langle c', E_i(c)\rangle$, $b_i = \langle h', E_i(h)\rangle$. Since $\tilde{J}(c') = -a_2E_1(c) + a_1E_2(c)$ and $\|\tilde{J}(c')\| = \|h'\|$, we then have

$$\langle D_{c'}c', \xi\rangle = \frac{a_1'a_2 - a_1a_2'}{\|h'\|}. \tag{4.3}$$

Furthermore

$$\begin{cases} Ad(exp(\theta e_3))e_1 = (\cos 2\theta)e_1 + (\sin 2\theta)e_2 \\ Ad(exp(\theta e_3))e_2 = -(\sin 2\theta)e_1 + (\cos 2\theta)e_2 \end{cases} \tag{4.4}$$

and thus

$$\begin{cases} \{R_{exp(\theta e_3)}\}_* E_1(h) = (\cos 2\theta)E_1(c) - (\sin 2\theta)E_2(c) \\ \{R_{exp(\theta e_3)}\}_* E_2(h) = (\sin 2\theta)E_1(c) + (\cos 2\theta)E_2(c) \end{cases}. \tag{4.5}$$

Thus by (4.2)

$$\begin{cases} a_1 = b_1\cos 2\theta + b_2\sin 2\theta \\ a_2 = -b_1\sin 2\theta + b_2\cos 2\theta \\ a_3 = \theta' \end{cases} \tag{4.6}$$

Since h' is horizontal, it follows that $\|h'\|^2 = b_1^2 + b_2^2$. So by (4.3) and (4.6)

$$\langle D_{c'}c', \xi\rangle = 2\theta'\|h'\| + \frac{b_1'b_2 - b_1b_2'}{\|h'\|}.$$

By lemmas 4.3.3, 4.3.7 and proposition 4.3.6 we then have that

$$\begin{aligned} k_g\|\gamma'\|^3 &= \langle D_{\gamma'}\gamma', J(\gamma')\rangle = \langle p_*(D_{h'}h',), p_* \circ \tilde{J}(h')\rangle \\ &= 4\langle D_{h'}h', \tilde{J}(h')\rangle = -4(b_1'b_2 - b_1b_2'). \end{aligned}$$

Since $\|\gamma'\| = 2\|h'\|$, the result follows. $\square$

Remark. By lemma 4.3.9 it follows that there exists an asymptotic lift of γ. Also note that if c_1 and c_2 are asymptotic lifts of γ then $c_2 = R_a(c_1)$, $a \in \mathbb{S}^1$.

Lemma 4.3.10. *Let $c(t) = p(t) \cdot exp\{\theta(t)e_3\}$ be an asymptotic lift of a curve $\gamma : \mathbb{R} \to \mathbb{S}^2$ where p is a horizontal lift of γ, $\xi = -\dfrac{\tilde{J}(c')}{\|\tilde{J}(c')\|}$ and $\alpha(t)$ the angle between $c'(t)$ and E_3 such that $0 < \alpha(t) < \pi$. Then*

1. $\|c'\| \cos\alpha = \frac{k_g\|\gamma'\|}{2}$, $\|c'\| \sin\alpha = \frac{\|\gamma'\|}{2}$.
2. *The restriction of ξ to c is left invariant.*
3. *If $\|\gamma'\|^2(1 + k_g^2) = 4$, then $\|c'\| = 1$ and $D_{c'}c' = \alpha'(c' \times \xi)$.*

Proof.

1. $c' = \{R_{exp(\theta e_3)}\}_* h' + \theta' E_3(c)$. Thus $\langle c', E_3\rangle = \theta' = \|c'\|\|E_3\| \cos\alpha$ and thus $\|c'\| \cos\alpha = \theta' = \frac{k_g\|\gamma'\|}{2}$ since c is an asymptotic lift. Moreover, since $\|\tilde{J}(c')\| = \|h'\|$ $\|c'\| \sin\alpha = \|h'\|$.

2. Since c is an asymptotic curve, it follows that $\langle D_{c'}\xi, c'\rangle = 0$ and $\langle D_{c'}\xi, \xi\rangle = 0$. Thus there exists a real valued function λ on $\mathbb{R}$ such that $D_{c'}\xi = \lambda(c' \times \xi)$. Then

$$\begin{aligned}\langle \xi, c' \times E_3\rangle =& \langle \xi, D_{c'}E_3\rangle = -\langle D_{c'}\xi, E_3\rangle \\ =& \lambda\langle \xi \times c', E_3\rangle = \lambda\langle \xi, c' \times E_3\rangle.\end{aligned}$$

   Since $\langle \xi, c' \times E_3\rangle = \|c' \times E_3\| > 0$ it follows that $\lambda = 1$. We then claim that a vector field v is left invariant along c if and only if $D_{c'}v = c' \times v$. To see this set $f_i(t) = \langle v(t), E_i(c(t))\rangle$. Then $D_{c'}v = \sum_{i=1}^{3} f_i' E_i(c) + c' \times v$. Since E_i is left invariant, v will be left invariant if and only if f_i is constant for $i = 1, 2, 3$. Similarly we have that v is right invariant along c if and only if $D_{c'}v = v \times c'$.

3. Suppose that $\|\gamma'\|^2(1 + k_g^2) = 4$. Then $\|c'\| = 1$ by (1). Since $\langle D_{c'}c', c'\rangle = 0$ and $\langle D_{c'}c', \xi\rangle = 0$ there exists a real valued function μ on $\mathbb{R}$ such that $D_{c'}c' = \mu(c' \times \xi)$. Differentiating $\cos\alpha = \langle c', E_3\rangle$ yields

$$\begin{aligned}-\alpha' \sin\alpha =& \langle D_{c'}c', E_3\rangle + \langle c', D_{c'}E_3\rangle \\ =& \mu\langle c' \times \xi, E_3\rangle + \langle c', c' \times E_3\rangle \\ =& -\mu\langle \xi, c' \times E_3\rangle \\ =& -\mu\|c' \times E_3\|.\end{aligned}$$

Since $||c' \times E_3\| = \sin\alpha$, it follows that $\mu = \alpha'$.

$\square$

Let $U\mathbb{S}^2 = \{(x,y)| \|x\| = \|y\| = 1, \langle x, y\rangle = 0\} \subset \mathfrak{su}_2 \times \mathfrak{su}_2$ be the unit tangent bundle of $\mathbb{S}^2$. Define $h_1 : U\mathbb{S}^2 \to \mathbb{S}^2$ by $h_1(x,y) = x$ and $h_2 : \mathbb{S}^3 \to U\mathbb{S}^2$ by $h_2(a) = (Ad(a)e_3, Ad(a)e_1)$. Note that h_2 is a double covering and the Hopf map is $h = h_1 \circ h_2$. By lemma 4.3.9, given a horizontal lift of $\gamma : \mathbb{R} \to \mathbb{S}^2$ we can generate a corresponding asymptotic lift. We provide a characterization of such horizontal lifts.

Lemma 4.3.11. *$c'(t) \in H_{c(t)}$ if and only if $\langle c(t)^{-1} \cdot c'(t), e_3\rangle = 0$.*

Proof. Set $a(s) = c(t) \cdot exp\{s(c(t)^{-1} \cdot c'(t))\}$. Since $a(0) = c(t)$ and $a'(0) = c'(t)$ it follows that $a'(0) = c'(t)$. Hence $\langle c'(t), E_3\rangle = \langle a'(0), E_3\rangle = \langle c(t)^{-1}c'(t), e_3\rangle$. $\square$

In terms of the double cover h_2, c is an asymptotic lift if

Lemma 4.3.12. *Let $c : \mathbb{R} \to \mathbb{S}^3$ be a curve in $\mathbb{S}^3$ such that $h_2(c) = \frac{\gamma'}{\|\gamma'\|}$. Then c is an asymptotic lift of γ.*

Proof. Without loss of generality assume that $\|\gamma'\| = 1$. Since the unit tangent vector of γ is $T = (\gamma, \gamma')$ it then follows that:

$$Ad(c)e_3 = \gamma \tag{4.7}$$

$$Ad(c)e_1 = \gamma'. \tag{4.8}$$

By (4.7) there exists a real valued function θ on $\mathbb{R}$ such that $c(t) = h(t) \cdot exp\{\theta(t)e_3\}$ where h is a horizontal lift of γ. Then by lemma 4.3.9, it remains only to show that $2\theta' = k_g$. Recalling (4.4), it follows that

$$\begin{cases} Ad(exp(\theta e_3))e_1 = (\cos 2\theta)e_1 + (\sin 2\theta)e_2 \\ Ad(exp(\theta e_3))e_2 = -(\sin 2\theta)e_1 + (\cos 2\theta)e_2 \end{cases}$$

and hence

$$\begin{cases} Ad(c)e_1 = Ad(h)(\cos(2\theta)e_1 + \sin(2\theta)e_2) \\ Ad(c)e_2 = Ad(h)(-\sin(2\theta)e_1 + \cos(2\theta)e_2) \end{cases}$$

Thus from (4.8) it follows that

$$\begin{aligned}
\gamma'' =& 2\theta' Ad(c)e_2 + \cos(2\theta)(Ad(h)e_1)' + \sin(2\theta)(Ad(h)e_2)' \\
=& -2\theta' \sin(2\theta) Ad(h)e_1 + \cos(2\theta)(Ad(h)e_1)' \\
&+ 2\theta' \cos(2\theta) Ad(h)e_2 + \sin(2\theta)(Ad(h)e_2)' \\
=& cos(2\theta)(Ad(h)e_1)' + \sin(2\theta)(Ad(h)e_2)' \\
&+ 2\theta'[Ad(h)(\cos(2\theta)e_2 - \sin(2\theta)e_1)] \\
=& \cos(2\theta)(Ad(h)e_1)' + \sin(2\theta)(Ad(h)e_2)' + 2\theta' Ad(c)e_2.
\end{aligned}$$

Then the geodesic curvature of γ is given by

$$k_g = \frac{1}{2}\langle \gamma'', [\gamma, \gamma'] \rangle = \langle \gamma'', Ad(c)e_3 \rangle = 2\theta' + P,$$

where

$$\begin{aligned}
P =& \langle \cos(2\theta)(Ad(h)e_1)' + \sin(2\theta)(Ad(h)e_2)', Ad(c)e_2 \rangle \\
=& \cos^2(2\theta)\langle (Ad(h)e_1)', Ad(h)e_2 \rangle - \sin^2(2\theta)\langle (Ad(h)e_2)', Ad(h)e_1 \rangle \\
&- \cos(2\theta)\sin(2\theta)\langle (Ad(h)e_1)', Ad(h)e_1 \rangle + \cos(2\theta)\sin(2\theta)\langle (Ad(h)e_2)', Ad(h)e_2 \rangle.
\end{aligned}$$

Now

$$\begin{aligned}
\langle (Ad(h)e_1)', Ad(h)e_1 \rangle =& \langle h'e_1h^{-1} - he_1h^{-1}h'h^{-1}, he_1h^{-1} \rangle \\
=& -\frac{1}{2}\mathrm{tr}(h'e_1h^{-1}he_1h^{-1}) + \frac{1}{2}\mathrm{tr}(he_1h^{-1}h'h^{-1}he_1h^{-1}) \\
=& -\frac{1}{2}\mathrm{tr}(h'e_1^2h^{-1}) + \frac{1}{2}\mathrm{tr}(h'e_1^2h^{-1}) \\
=& \ 0.
\end{aligned}$$

Similarly $\langle (Ad(h)e_2)', Ad(h)e_2 \rangle = 0$. Then

$$\begin{aligned}
\langle (Ad(h)e_2)', Ad(h)e_1 \rangle =& \langle h'e_2h^{-1}, he_1h^{-1} \rangle - \langle he_2h^{-1}h'h^{-1}, he_1h^{-1} \rangle \\
=& \ \frac{1}{2}\mathrm{tr}(h'[e_1, e_2]h^{-1}) \\
=& \ \mathrm{tr}(h'e_3h^{-1}).
\end{aligned}$$

Similarly, $\langle (Ad(h)e_1)', Ad(h)e_2 \rangle = -\mathrm{tr}(h'e_3h^{-1})$. Thus

$$P = \cos^2(2\theta)(\mathrm{tr}(he_3h^{-1})) + \sin^2(2\theta)(\mathrm{tr}(h'e_3h^{-1}))$$

that is $P = 2\langle h^{-1}h', e_3\rangle = \langle (Ad(h)e_1)', Ad(h)e_2\rangle$. Thus by lemma 4.3.11, $P = 0$ and so $\theta' = \frac{k_g}{2}$. □

In particular if we wish to generate flat tori out of all possible flat surfaces in $\mathbb{S}^3$ it is necessary that γ be periodic. This leads to:

Theorem 4.3.13. *Let γ be a regular curve on $\mathbb{S}^2$ and let c be an asymptotic lift of γ. If γ is l-periodic, then c is $2l$-periodic.*

Proof. By remark 4.3 it follows that asymptotic lifts exist and are unique up to right translation and that $h_2(c) = \frac{\gamma'}{\|\gamma'\|}$. Thus $h_2(c)$ is l-periodic. Since h_2 is a double covering, c is thus $2l$-periodic. □

Let $a_i : \mathbb{R} \to \mathbb{S}^3$ be two curves, for $i = 1, 2$, such that
$a_i(0) = e$, $\|a_i'\| = 1$, $a_1'(0) \times a_2'(0) \neq 0$, where e is the identity of $\mathbb{S}^3$. Define
$f : \mathbb{R}^2 \to \mathbb{S}^3$ by $f(t_1, t_2) = a_1(t_1) \cdot a_2(t_2)$, and set

$$\begin{cases} \xi_0 = \dfrac{a_1'(0) \times a_2'(0)}{\|a_1'(0) \times a_2'(0)\|} \\ \xi_1(t) = (L_{a_1(t)})_* \xi_o, \xi_2(t) = \{R_{a_2(t)}\}_* \xi_0 \\ n_i = \xi_i \times a_i', k_i = \langle D_{a_i'} a_i', n_i\rangle \\ \omega(t_1, t_2) = \omega_0 - \int_0^{t_1} k_1(t)dt - \int_0^{t_2} k_2(t)dt \end{cases} \tag{4.9}$$

where ω_0 denotes the angle between $a_1'(0)$ and $a_2'(0)$ such that $0 < \omega_0 < \pi$.

Lemma 4.3.14. *If $0 < \omega < \pi$ and $\langle a_i', \xi_i\rangle = 0$ for $i = 1, 2$, then the map f is a flat asymptotic Tchebychev immersion such that in the first fundamental form $F = \cos\omega$ and in the second fundamental form $\sin\omega = m$.*

Proof. Let X_i and Y_i be vector fields along a_i given by

$$\begin{cases} X_i(t) = (L_{a_1(t)})_* a_i'(0), \;\; X_2(t) = (R_{a_2(t)})_* a_2'(0) \\ Y_1(t) = (L_{a_1(t)})_* n_1(0), \;\; Y_2(t) = (R_{a_2(t)})_* n_2(0) \end{cases} \tag{4.10}$$

Since $\xi_i = X_i \times Y_i$ and $\langle a_i', \xi_i\rangle = 0$ there exists a real valued function Ω_i such that

$$a_i' = (\cos\Omega_i)X_i + (\sin\Omega_i)Y_i, \;\; \Omega_i(0) = 0. \tag{4.11}$$

So $n_i = -(\sin\Omega_i)X_i + (\cos\Omega_i)Y_i$. Then

$$\begin{aligned} D_{a_i'}a_i' &= \Omega_i' n_i + (\cos\Omega_i)D_{a_i'}X_i + (\sin\Omega_i)D_{a_i'}Y_i \\ &= \Omega_i' n_i \pm a_i' \times ((\cos\Omega_i)X_i + (\sin\Omega_i)Y_i) \\ &= \Omega_i' n_i. \end{aligned}$$

It follows from $\kappa_i = \langle D_{a_i'}a_i', n_i\rangle$ that $\kappa_i = \Omega_i'$ and thus

$$\omega(t_1,t_2) = \omega_0 - \Omega_1(t_1) + \Omega_2(t_2). \tag{4.12}$$

By (4.10) and (4.11) we have

$$f_i(t_1,t_2) = (\cos\Omega_i(t_i))\Phi_* a_i'(0) + (\sin\Omega_i(t_i))\Phi_* n_i(0).$$

where $f_i = \dfrac{\partial f}{\partial t_i}$, $\Phi = L_{a_1(t_1)} \circ R_{a_2(t_2)}$. So it follows from (4.12) that the angle between f_1 and f_2 is $\omega(t_1,t_2)$. By the assumption that $0 < \omega < \pi$, f is an immersion with $E = G = 1$ and $F = \cos\omega$ in the first fundamental form. Define ξ along f by

$$\xi(t_1,t_2) = \{L_{a_1} \circ R_{a_2}\}_* \xi_0$$

Then since ξ is left invariant along f_1 and right invariant along f_2, we have

$$D_{f_1}\xi = f_1 \times \xi, \; D_{f_2}\xi = \xi \times f_2.$$

Since $\langle f_i(t_1,t_2), \xi(t_1,t_2)\rangle = \langle a_i'(t_i), \xi_i(t_i)\rangle = 0$ we have

$$\xi = \frac{f_1 \times f_2}{\|f_1 \times f_2\|}.$$

Therefore in the second fundamental form $l = m = 0$ and
$n = -\langle D_{f_1}\xi, f_2\rangle = \langle \xi, f_1 \times f_2\rangle = \sin\omega$. □

We are thus at last ready to enumerate the properties that pairs of curves on $\mathbb{S}^2$ must satisfy in order to generate flat surfaces in $\mathbb{S}^3$.

Definition 4.3.15. For $i = 1, 2$, let $\gamma_i : \mathbb{R} \to \mathbb{S}^2$ be a regular curve on $\mathbb{S}^2$. The pair $\Gamma = (\gamma_1, \gamma_2)$ is said to be admissible if

$$\gamma_i(0) = e_3, \frac{\gamma'(0)}{\|\gamma'(0)\|} = e_1 \tag{4.13}$$

$$\|\gamma_i'\|^2(1 + k_i^2) = 4 \tag{4.14}$$

$$k_1(t_1) > k_2(t_2) \text{ for all } (t_1, t_2) \in \mathbb{R}^2 \tag{4.15}$$

where k_i is the geodesic curvature of γ_i.

Theorem 4.3.16. *For an admissible pair Γ, let c_i be the asymptotic lift of γ_i to $\mathbb{S}^3$ with $c_i(0) = \mathbb{1}$ and define $f_\Gamma(t_1, t_2) = c_1(t_1) \cdot c_2(t_2)^{-1}$. Then the map f_Γ is a flat asymptotic Tchebychev immersion.*

Proof. Let $\alpha_i(t)$ be the angle between $c_i'(t)$ and E_3 such that $0 < \alpha_i(t) < \pi$. Then it follows from lemma 4.3.10 that $\|c_i'\| = 1$ and $\cot\alpha_i = k_i$. By (4.15)

$$0 < \alpha_2(t_2) - \alpha_1(t_1) < \pi. \tag{4.16}$$

Set $a_1(t) = c_1(t)$ and $a_2(t) = c_2(t)^{-1}$. Then it follows that, setting $F = f_\Gamma$, $F(t_1, t_2) = a_1(t_1) \cdot a_2(t_2)$, $a_i(0) = e$, $\|a_i'\| = 1$. Let $c_i'(0) = x_1 E_1(e) + x_2 E_2(e) + x_3 E_3(e)$, then by lemma 4.3.10 $\|c'\| \cos\alpha = \theta'$ and $\|c' \sin\alpha = \|h'\|$. Thus $x_3 = \cos\alpha$ and $x_2 = \sin\alpha$ and since $\|c'\| = 1$ it follows that $x = 0$. Then by (4.13)

$$c_i'(0) = \{\sin\alpha_i(0)\}E_2(e) + \{\cos\alpha_i(0)\}E_3(e).$$

Moreover $\langle a_1'(0), a_2'(0)\rangle = -\cos(\alpha_1 - \alpha_2)$ and since $\cos\omega_0 = \langle a_1'(0), a_2'(0)\rangle$ it follows that $\omega_0 = \pi - \alpha_2(0) + \alpha_1(0)$ hence $a_1'(0) \times a_2'(0) \neq 0$ by (4.16). Define ξ_0, ξ_i, n_i, k_i and ω as in (4.9), then

$$\xi_0 = -\frac{\tilde{J}(c_i'(0))}{\|\tilde{J}(c_i'(0))\|}$$

and by lemma 4.3.10 it follows that

$$\xi_1 = -\frac{\tilde{J}(c_1')}{\|\tilde{J}(c_1')\|}, \xi_2 = \tau_*\left(\frac{\tilde{J}(c_2')}{\|\tilde{J}(c_2')\|}\right) \tag{4.17}$$

where $\tau_*(a) = a^{-1}$. Note also that

$$\begin{aligned} D_{a_i'} a_i' =& -\alpha_i'(\xi_i \times a_i') \\ =& \alpha_i' n_i \end{aligned}$$

and thus $k_i = -\alpha_i'$. Hence $\omega(t_1, t_2) = \pi + \alpha_1(t_1) - \alpha_2(t_2)$ and so $0 < \omega < \pi$ by (4.16). Since $\langle a_i', \xi_i\rangle = 0$ by (4.17), lemma 4.3.14 implies that f_Γ is a flat asymptotic Tchebychev immersion. $\square$

Conversely we can generate a regular curve on $\mathbb{S}^2$ from an asymptotic curve in $\mathbb{S}^3$.

Lemma 4.3.17. *Let α be a real-valued function such that $0 < \alpha(t) < \pi$ and $c : \mathbb{R} \to \mathbb{S}^3$ a curve with $c(0) = e$, $\|c'(0) \times E_3\| = \sin\alpha(0)$ and $\|c'\| = 1$. Suppose that there exists a vector field ξ along c such that*

$$\xi(0) = \frac{c'(0) \times E_3}{\|c'(0) \times E_3\|},$$

$\langle c', \xi\rangle = 0, D_{c'}c' = \alpha'(c' \times \xi)$ and $D_{c'}\xi = c' \times \xi$. Then $\gamma = h \circ c$ is a regular curve and c is an asymptotic lift of γ. Furthermore $\|\gamma'\| = 2\sin\alpha$, and the geodesic curvature k of γ satisfies $k = \cot\alpha$.

Proof. Set $a_1(t) = c(t)$ and $a_2(t) = exp(te_3)$. Then $a_i(0) = e, \|a_i'\| = 1$ and $a_i'(0) \times a_2'(0) \neq 0$. Define $\xi_0, \xi_i(t), n_i(t), k_i(t)$ and $\omega(t_1, t_2)$ by (4.9). Since $\xi(0) = \xi_0$ and ξ is left invariant along c it follows that $\xi_1(t) = \xi(t)$ and so $\langle a_1', xi_1\rangle = \langle c', \xi\rangle = 0$. Since E_3 is invariant under the action of $\mathbb{S}^1$ it follows that $\langle a_2'(t), \xi_2(t)\rangle = \langle E_3, \xi_0\rangle = 0$. Thus

$$\langle a_i', \xi_i\rangle = 0. \tag{4.18}$$

It follows that $k_1 = \langle D_{c'}c', \xi \times c'\rangle = -\alpha'$ and $k_2 = 0$. By the assumption that $\sin\alpha(0) = \|c'(0) \times E_3\|$ the angle ω_0 between $a_1'(0)$ and $a_2'(0)$ is equal to $\alpha(0)$. Thus

$$\omega(t_1, t_2) = \alpha(0) + \int_0^{t_1} \alpha'(t)dt = \alpha(t_1).$$

Since $0 < \alpha(t) < \pi$ it follows that

$$0 < \omega(t_1, t_2) < \pi. \tag{4.19}$$

Define $f : \mathbb{R}^2 \to \mathbb{S}^3$ by $f(t_1, t_2) = a_1(t_1) \cdot a_2(t_2)$. By (4.18) and (4.19) it follows by lemma 4.3.14 that f is a flat asymptotic Tchebychev immersion and that the first fundamental form $F = \cos\omega$ and in the second fundamental form $m = \sin\omega$. Since $f(t_1, t_2) = c(t_1) \cdot exp(t_2e_3)$ and $\omega(t_1, t_2) = \alpha(t_1)$, the angle between $c'(t)$ and E_3 is equal to $\alpha(t)$. Hence by lemma 4.3.3 it follows that $\|\gamma'\| = \|p_*c'\| = 2\|c'\|\sin\alpha = 2\sin\alpha > 0$ and so γ is regular. Since in the second fundamental form $l = 0$, we have that c is an asymptotic lift of γ. By lemma 4.3.10 $k = \cot\alpha$. □

We can also write the flat asymptotic Tchebychev immersion in terms of the asymptotic curves by:

Lemma 4.3.18. *Let f be a flat asymptotic Tchebychev immersion. Then $f(t_1, t_2) = f(t_1, 0) \cdot f(0, 0)^{-1} \cdot f(0, t_2)$ for all $(t_1, t_2) \in \mathbb{R}^2$.*

Proof. A calculation shows that the first and second fundamental forms of $f(t_1, t_2) = f(t_1, 0) \cdot f(0, 0)^{-1} \cdot f(0, t_2)$ are given by

$$I = du^2 + 2\cos\omega du dv + dv^2$$
$$II = 2\sin\omega du dv$$

where ω is the angle between the tangent vector to the curves $f(t_1, 0)$ and $f(0, t_2)$. These agree with the first and second fundamental forms of the flat asymptotic Tchebychev immersion f and thus by the fundamental theorem of surfaces the result follows. □

We now show the main result for flat surfaces from the Kitagawa approach: given a flat asymptotic Tchebychev immersion of a flat surface Σ in $\mathbb{S}^3$ there exists up to isometry an admissible pair that generates it.

Theorem 4.3.19. *Let $f : \mathbb{R}^2 \to \mathbb{S}^3$ be a flat asymptotic Tchebychev immersion. If the mean curvature of f is bounded, then there exists an admissible pair Γ such that $f_\Gamma = \Phi \circ f$ for some isometry Φ of $\mathbb{S}^3$.*

Proof. Set

$$f_i = \frac{\partial f}{\partial t_i}, \ \xi = \frac{f_1 \times f_2}{\|f_1 \times f_2\|}$$

and $g_{ij} = \langle f_i, f_j \rangle, \tau(a) = a^{-1}$ and $h_{ij} = \langle D_{f_i} f_j, \xi \rangle$. Replacing f by $F \circ \tau$ if necessary, it is possible to assume that $h_{12} > 0$. Moreover there exists a function ω on $\mathbb{R}^2$ such that the first fundamental form has the format

$$du^2 + 2\cos\omega du dv + dv^2$$

while the second fundamental form satisfies

$$2\sin\omega du dv$$

with $\omega(t_1, t_2) = \omega_1(t_1) + \omega_2(t_2)$. Moreover the mean curvature H of f satisfies $H = -\cot\omega$. Since H is bounded there exists $\delta > 0$ such that $\delta \le \omega \le \pi - \delta$. Replacing f by $\Phi \circ f$ for some orientation preserving isometry Φ of $\mathbb{S}^3$, it is then possible to assume that

$$f(0,0) = e, E_1(e) = \frac{f_1(0,0) \times E_3}{\sin \omega_1(0)} = \frac{E_3 \times f_2(0,0)}{\sin \omega_2(0)}. \tag{4.20}$$

Then by lemma 4.3.18 it follows that

$$f(t_1, t_2) = f(t_1, 0) \cdot f(0, t_2). \tag{4.21}$$

Set

$$c_1(t) = f(t,0), \xi_1(t) = \xi(t,0), \alpha_1(t) = \omega_1(t)$$

$$c_2(t) = \tau \circ f(0,t), \xi_2(t) = -\tau_* \xi(0,t), \alpha_2(t) = \pi - \omega_2(t).$$

Then

$$\begin{cases} c_i(0) = e, \|c_i'\| = 1, \|c_i'(0) \times E_3\| = \sin \alpha_i(0) \\ \xi_i(0) = \dfrac{c_i'(0) \times E_3}{\|c_i'(0) \times E_3\|} \\ 0 < \alpha_i < \pi, \langle c_i', \xi_i \rangle = 0 \end{cases} \tag{4.22}$$

Then

$$D_{c_i'} \xi_i = c_i' \times \xi_i. \tag{4.23}$$

By Frenet-Serret it follows that

$$D_{f_1} f_1 = k_1 n_1 = \omega_1'(F_1 \times \xi)$$

and

$$D_{f_2} f_2 = k_2 n_2 = \omega_2'(\xi \times F_2).$$

Hence

$$D_{c_i'} c_i' = \alpha_i'(c' \times \xi_i). \tag{4.24}$$

Then by lemma 4.3.17 it follows that γ_i is a regular curve and that c_i is an asymptotic lift of γ_i with

$$\|\gamma_i'\| = 2 \sin \alpha_i, k_i = \cot \alpha_i.$$

Hence $\|\gamma_i'\|^2(1 + k_i^2) = 4$ and $k_1 > k_2$ since $\alpha_2 - \alpha_1 = \pi - \omega > 0$. By (4.20) it follows that $\gamma_i(0) = e_3$ and

$$\frac{\gamma_i'(0)}{\|\gamma_i'(0)\|} = e_1.$$

Hence $\Gamma = (\gamma_1, \gamma_2)$ is an admissible pair. Then by (4.21)

$f_\Gamma(t_1, t_2) = c_1(t_1, t_2)^{-1} = f(t_1, t_2)$. □

Thus it remains to give criteria that ensures that we select only those admissible pairs that generate flat tori out of all possible flat surfaces in $\mathbb{S}^3$. We show that the necessary and sufficient condition is that the admissible pairs be periodic.

4.3.1 Periodic Admissible Pairs

Definition 4.3.20. Let $\Gamma = (\gamma_1, \gamma_2)$ be an admissible pair. Then Γ is said to be periodic if there exists positive numbers l_1 and l_2 such that γ_i is l_i periodic.

Define the group $G(\Gamma)$ by

$$G(\Gamma) = \{p \in Diff(\mathbb{R}^2) | f_\Gamma \circ p = f_\Gamma\}$$

where $f_\Gamma(t_1, t_2) = c_1(t_1) \circ c_2(t_2)^{-1}$ is a flat asymptotic Tchebychev immersion given in terms of the asymptotic lifts c_i of γ_i. We will show that when Γ is periodic $\mathbb{R}^2/G(\Gamma)$ is compact and thus the resultant surface is a flat torus. We can identify $G(\Gamma)$ with an additive subgroup of $\mathbb{R}^2$ as a result of

Theorem 4.3.21. *Let f be a flat asymptotic Tchebychev immersion and $p : \mathbb{R}^2 \to \mathbb{R}^2$ a diffeomorphism such that $f \circ p = f$. If $p(0,0) = (r_1, r_2)$ then $p(t_1, t_2) = (t_1 + r_1, t_2 + r_2)$ for all $(t_1, t_2) \in \mathbb{R}^2$.*

Proof. We prove the result by showing that

$$\frac{\partial p}{\partial t_i} = (\delta_{1j}, \delta_{2j}). \tag{4.25}$$

Let $(x_1, x_2) \in \mathbb{R}^2$ and let $p(x_1, x_2) = (y_1, y_2)$. The the curve $t \mapsto p(x_1 + t, x_2)$ is a unit speed asymptotic curve on $\mathbb{R}^2$ starting at (y_1, y_2). We note that since $f \circ p = f$, $f^* \circ p^* = f^*$ and thus p must be a non-rotational isometry since it preserves the lattice. Thus the following four cases may occur

$$p(x_1 + t, x_2) = (y_1 + t, y_2), \tag{4.26}$$
$$p(x_1 + t, x_2) = (y_1 - t, y_2), \tag{4.27}$$
$$p(x_1 + t, x_2) = (y_1, y_2 + t), \tag{4.28}$$
$$p(x_1 + t, x_2) = (y_1, y_2 - t). \tag{4.29}$$

We will show that only the first case is possible. Assume that

$p(x_1+t,x_2)=(y_1-t,y_2)$. Then it follows that

$$f(x_1+t,x_2)=f(y_1-t,y_2). \tag{4.30}$$

In particular we have that

$$f\left(\frac{x_1+y_1}{2},x_2\right)=f\left(\frac{x_1+y_1}{2},y_2\right)$$

since if $t=\dfrac{y_1-x_1}{2}$ $f\circ p(x_1+t,x_2)=f(x_1+t,x_2)$, that is

$$f\left(\frac{x_1+y_1}{2},x_2\right)=f\left(y_1-t,y_2\right)=f\left(\frac{x_1+y_1}{2},y_2\right).$$

Hence by lemma (4.3.18) we have

$$f(t,x_2)=f(t,y_2)$$

since

$$f\left(\frac{x_1+y_1}{2},x_2\right)=f\left(\frac{x_1+y_1}{2},0\right)f(0,x_2)=f\left(\frac{x_1+y_1}{2},y_2\right)$$

which implies that $f(0,x_2)=f(0,y_2)$. Then it follows from (4.30) that $f(x_1+t,x_2)=f(y_1-t,x_2)$. Differentiating at $t=\dfrac{y_1-x_1}{2}$ we have

$$f_1\left(\frac{x_1+y_1}{2},x_2\right)=0$$

where f_1 is the partial derivative with respect to the first variable. This is a contradiction since f is a flat asymptotic Tchebychev immersion with $\langle f_1,f_1\rangle=1$. Assume next that $p(x_1+t,x_2)=(y_1,y_2+t)$. Set $c(t)=f(x_1+t,x_2)$, $v(t)=\xi(x_1+t,x_2)$ where $\xi=\dfrac{f_1\times f_2}{\|f_1\times f_2\|}$. Since f is a flat asymptotic Tchebychev immersion we have

$$D_{f_1}\xi=f_1\times\xi \text{ and } D_{f_2}\xi=\xi\times f_2 \tag{4.31}$$

and since ξ satisfies $D_{f_i}\xi=\tau_i n_i$ where τ_i is the second curvature of the two asymptotic curves, it follows from $\tau_1=-\tau_2$ that $D_{c'}v=c'\times v$. Moreover since $\xi\circ p=\pm\xi$ it follows from (4.28) that

$$c(t)=f(y_1,y_2+t),v(t)=\pm\xi(y_1,y_2+t).$$

Then by (4.31) $D_{c'}v=v\times c'$. Hence $c'\times v=v\times c'=0$ which is a contradiction.

Similarly (4.29) cannot hold and thus (4.26) holds. In particular (4.25) holds for $j = 1$ and similarly holds for $j = 2$. $\square$

Moreover $G(\Gamma)$ is a discrete subgroup of $\mathbb{R}^2$ and so $\mathbb{R}^2/G(\Gamma)$ is a two dimensional manifold. Let $\pi : \mathbb{R}^2 \to \mathbb{R}^2/G(\Gamma)$ be the canonical projection. Then the immersion f_Γ induces a flat immersion $\bar{f}_\Gamma : \mathbb{R}^2\backslash G(\Gamma) \to \mathbb{S}^3$ such that $\bar{f}_\Gamma \circ \pi = f_\Gamma$. In order to prove the result we require some lemmas. We will show that f_Γ is periodic if and only if the curves γ_i whose lifts are the asymptotic curves c_i, are periodic. We do this in three parts. Let c_i be the asymptotic lift of γ_i such that $c_i(0) = e$ and set $f = f_\Gamma$, $\xi = f_1 \times_2 \backslash \|f_1 \times f_2\|$. Then

$$Ad(c_i)e_3 = \gamma_i, \quad Ad(c_i)e_1 = \frac{\gamma_i'}{\|\gamma_i'\|} \tag{4.32}$$

$$f(t_1, t_2) = c_1(t_1) \cdot c_2(t_2)^{-1}. \tag{4.33}$$

We first show that at repeated values of the immersion where ξ is also repeated that the curves γ_1 and γ_2 coincide and have the same unit tangent vector.

Lemma 4.3.22. *Let $(l_1, l_2) \in \mathbb{R}^2$. If $f(l_1, l_2) = f(0, 0)$ and $\xi(l_1, l_2) = \xi(0, 0)$, then $\gamma_1(l_1) = \gamma_2(l_2)$ and $e_1 = \dfrac{\gamma_i'(l_i)}{\|\gamma_i'(l_i)\|}$ for $i = 1, 2$.*

Proof. We may assume without loss of generality that $f(l_1, l_2) = f(0, 0) = e$. Then (4.33) implies that $c_1(l_1) = c_2(l_2)$ and so $\gamma_1(l_1) = \gamma_2(l_2)$ by (4.32). Replacing f by $f \circ \phi$ if necessary, where $\phi : \mathbb{S}^3 \to \mathbb{S}^3$ is the map $\phi(a) = a^{-1}$, we may assume that m in the second fundamental form satisfies $m > 0$. Moreover since $D_{f_1}\xi = f_1 \times \xi$ and $D_{f_2}\xi = \xi \times f_2$ (by (4.31)) we have that ξ is left invariant along $t_1 \mapsto f(t_1, t_2)$ and right invariant along $t_2 \mapsto f(t_1, t_2)$. Thus

$$\xi(l_1, l_2) = \{R_{c_2(l_2)^{-1}}\}_*\{L_{c_1(l_1)}\}_*\xi(0, 0).$$

Since $\xi(0, 0) = E_1(e)$ and $c_1(l_1) = c_2(l_2)$ it follows that

$$\begin{aligned} E_1(e) = \xi(l_1, l_2) &= \{R_{c_2(l_2)^{-1}}\}_* E_1(c_1(l_1)) \\ &= \frac{d}{dt}\{c_1(l_1) \cdot exp(te_1) \cdot c_2(l_2)^{-1}\}|_{t=0} \\ &= \frac{d}{dt}\{c_i(l_i) \cdot exp(te_1) \cdot c_i(l_i)^{-1}\}|_{t=0} \\ &= \frac{d}{dt} exp\{tAd(c_i(l_i))e_1\}|_{t=0}. \end{aligned}$$

Thus $e_1 = Ad(c_i(l_i))e_1$ and so

$$e_1 = \frac{\gamma_i'(l_i)}{\|\gamma_i'(l_i)\|}$$

by (4.32) $\square$

Next we show that if f is periodic with period (l_1, l_2) the $\gamma_i(t + l_i)$ is the composition of a linear, orientation preserving isometry with $\gamma_i(l_i)$.

Lemma 4.3.23. *Let $(l_1, l_2) \in \mathbb{R}^2$ and let f satisfy*

$$f(t_1 + l_1, t_2 + l_2) = f(t_1, t_2) \tag{4.34}$$

for all $(t_1, t_2) \in \mathbb{R}^2$. Then there exists an orientation preserving linear isometry ϕ_i of $\mathfrak{su}_2$ such that $\gamma_i(t + l_i) = \phi_i(\gamma_i(t))$.

Proof. Let $\alpha_i(t)$ denote the angle between $c_i'(t)$ and E_3 such that $0 < \alpha_i(t) < \pi$ and let $\omega(t_1, t_2) = \pi + \alpha_1(t_1) - \alpha_2(t_2)$. Then recall that in the first fundamental form we have

$$\langle f_1, f_2 \rangle = \cos \omega, \quad 0 < \omega < \pi.$$

Then from (4.34) it follows that $\omega(t_1 + l_1, t_2 + l_2) = \omega(t_1, t_2)$ for all $(t_1, t_2) \in \mathbb{R}^2$. Thus

$$\alpha_1(t_1 + l_1) - \alpha_1(t_1) = \alpha_2(t_2 + l_2) - \alpha_2(t_2) = \mathit{constant}.$$

Since α_i is bounded, α_i is l_i-periodic. So it follows from lemma 4.3.10 that $\|\gamma_i'\|$ and k_i are l_i-periodic where k_i is the geodesic curvature of γ_i and thus the result follows. $\square$

Moreover we can show that if f is periodic with period (l_1, l_2) then the tangent vector to γ_i is periodic with period l_i.

Lemma 4.3.24. *Let $(l_1, l_2) \in \mathbb{R}^2$. If f is periodic with period (l_1, l_2) then $\gamma'(t + l_i) = \gamma_i'(t)$ for all $t \in \mathbb{R}$.*

Proof. Let $(s_1, s_2) \in \mathbb{R}^2$ and let $\phi_i = Ad(c_i(s_i)^{-1})$. Then by (4.32) we obtain

$$\phi_i(\gamma_i(s_i)) = e_3, \quad \phi_i(\gamma_i'(s_i) \backslash \|\gamma_i'(s_i)\|) = e_1. \tag{4.35}$$

Let $\tilde{\gamma}_i$ be a regular curve in $\mathbb{S}^2$ defined by

$$\tilde{\gamma}_i(t) = \phi_i(\gamma_i(t + s_i)).$$

Then

$$Ad(\tilde{c}_i)e_3 = \tilde{\gamma}_i, \quad Ad(\tilde{c}_i)e_1 = \frac{\tilde{\gamma}_i'}{\|\tilde{\gamma}_i'\|}.$$

Hence lemma 4.3.12 implies that $\tilde{c}_i$ is an asymptotic lift of $\tilde{\gamma}_i$. Since $\tilde{c}_i(0) = e$, it follows that $f_{\tilde{\Gamma}} = \tilde{c}_1(t_1) \cdot \tilde{c}_2(t_2)^{-1}$. Set $\tilde{f} = f_{\tilde{\Gamma}}$ and $\tilde{\xi} = \tilde{f}_1 \times \tilde{f}_2 \backslash \|\tilde{f}_1 \times \tilde{f}_2\|$. By (4.33) we obtain

$$\tilde{f}(t_1, t_2) = c_1(s_1)^{-1} \cdot f(t_1 + s_1, t_2 + s_2) \cdot c_2(s_2).$$

So it follows from (4.34) that $\tilde{f}(t_1 + l_1, t_2 + l_2) = \tilde{f}(t_1, t_2)$. In particular $\tilde{f}(l_1, l_2) = \tilde{f}(0, 0)$ and $\tilde{\xi}(l_1, l_2) = \tilde{\xi}(0, 0)$. Hence lemma 4.3.22 implies that

$$-\frac{\tilde{\gamma}_i'(l_i)}{\|\tilde{\gamma}_i'(l_i)\|} = e_1. \tag{4.36}$$

Since $\tilde{\gamma}_i'(l_i) = \phi_i(\gamma_i'(l_i + s_i))$, it follows from (4.32) and (4.36) that

$$\begin{aligned} \gamma_i'(l_i + s_i) &= Ad(c_i(s_i))\tilde{\gamma}_i'(l_i) = \|\tilde{\gamma}_i'(l_i)\| Ad(c_i(s_i))e_1 \\ &= \frac{\|\gamma_i'(l_i + s_i)\| \gamma_i'(s_i)}{\|\gamma_i'(s_i)\|}. \end{aligned}$$

By lemma 4.3.23 we have $\|\gamma_i'(l_i + s_i)\| = \|\gamma_i'(s_i)\|$. Hence $\gamma_i'(l_i + s_i) = \gamma_i'(s_i)$. □

Thus with these three lemmas we can now prove

Proposition 4.3.25. *Let* $(l_1, l_2) \in \mathbb{R}^2$. *If* f *satisfies* (4.34), *then* γ_i *is* l_i*-periodic.*

Proof. By lemma 4.3.23 there exists an orientation preserving linear isometry ϕ_i of $\mathfrak{su}_2$ such that

$$\gamma_i(t + l_i) = \phi_i(\gamma_i(t)). \tag{4.37}$$

Differentiating (4.37) we have $\gamma_i'(t + l_i) = \phi_i(\gamma_i(t))$. Hence lemma 4.3.24 implies that

$$\gamma_i'(t) = \phi_i(\gamma_i'(t)). \tag{4.38}$$

Since γ_i is a regular curve on $\mathbb{S}^2$, there exists $s_i \in \mathbb{R}$ such that $\gamma_i'(0)$ and $\gamma_i'(s_i)$ are linearly independent in $\mathfrak{su}_2$. So it follows from (4.38) that ϕ_i must be the identity. Then (4.37) implies that γ_i is l_i-periodic. □

Thus at least we are able to prove the main result of this section [23].

Theorem 4.3.26. *The quotient space* $\mathbb{R}^2/G(\Gamma)$ *is compact if and only if* Γ *is periodic.*

Proof. Assume that $\mathbb{R}^2/G(\Gamma)$ is compact. Then there exists positive numbers l_1 and l_2 such that $(l_1, l_2) \in G(\Gamma)$. Then f satisfies (4.34) and so by proposition 4.3.25 Γ is periodic. Conversely suppose that Γ is periodic. Then there exists positive numbers l_1 and l_2 such that γ_i is l_i-periodic. By Theorem 4.3.13 c_i is $2l_i$-periodic. Then (4.33) implies that

$$f(t_1 + 2l_1, t_2) = f(t_1, t_2 + 2l_2) = f(t_1, t_2).$$

So the group $G(\Gamma)$ contains $(2l_1, 0)$ and $(0, 2l_2)$. Hence $\mathbb{R}^2/G(\Gamma)$ is compact $\square$

Chapter 5

Finite Gap Curves

This chapter deals with finite gap curves. We introduce the notion of the complex curvature of a curve and develop the concept of finite gap in terms of frames, polynomial Killing fields and its spectral curve. We also provide conditions that ensure the closure of the curve along with restrictions under which a finite gap curve is spherical.

5.1 The Double Cover

Finite gap curves in $\mathbb{S}^3$ or $\mathbb{S}^2$ are curves whose complex curvature function $q(s) = \frac{1}{2}\kappa(s)e^{i\int \tau(x)dx}$ (for curves in $\mathbb{S}^3$) or curvature function κ (for curves in $\mathbb{S}^2$) satisfy that all flows of sufficiently high order (in NLS for curves on $\mathbb{S}^3$ and in mKdV for curves on $\mathbb{S}^2$) restricted to these curves to be linear combinations of lower order flows. We develop the theory of these curves in terms of their frames and spectral curves. Let

$$E_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; E_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

be the standard basis for $\mathbb{R}^3$ and let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be an arclength parameterized curve. Then the Frenet frame for γ is the map $G : \mathbb{R} \to SO_3$ satisfying

$$G' = G \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \tag{5.1}$$

where κ is the curvature of γ and τ its torsion. The curve γ can be recovered from κ and τ by solving (5.1) and

$$\gamma' = GE_0. \tag{5.2}$$

To extend this we make use of a double cover of SO_3. Let e_0, e_1,e_2 be the basis for $\mathfrak{su}_2$

$$e_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The correspondence

$$E_l \leftrightarrow e_l, l = 0, 1, 2 \tag{5.3}$$

extended linearly, is a vector space isomorphism $\mathbb{R}^3 \leftrightarrow \mathfrak{su}_2$. More explicitly

$$a_0E_0 + a_1E_1 + a_2E_2 \leftrightarrow a_0e_0 + a_1e_1 + a_2e_2 \tag{5.4}$$

or, in matrix form,

$$\begin{pmatrix} x_0, & x_1, & x_2 \end{pmatrix} \leftrightarrow \begin{pmatrix} ix_0 & x_1 + ix_2 \\ -x_1 + ix_2 & -ix_0 \end{pmatrix}. \tag{5.5}$$

These choices induce a double cover $SU_2 \to SO_3$ as follows: given $F \in SU_2$ the corresponding $G \in SO_3$ is given by

$$GE_i \leftrightarrow Fe_iF^{-1}, \text{ for all } i = 0, 1, 2 \tag{5.6}$$

where the arrow denotes the correspondence $\mathbb{R}^3 \leftrightarrow \mathfrak{su}_2$. Conversely, given $G \in SO_3$, (5.6) determines $F \in SU_2$ uniquely up to sign. Explicitly, for $F \in SU_2$,

$$F = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \; a = a_0 + ia_1, b = b_0 + ib_1$$

$$Fe_oF^{-1} = \begin{pmatrix} i(a\bar{a} - b\bar{b}) & -2iab \\ -2i\bar{a}\bar{b} & -i(a\bar{a} - b\bar{b}) \end{pmatrix}$$

$$Fe_1F^{-1} = \begin{pmatrix} -i(a\bar{b} + \bar{a}b) & i(b^2 - a^2) \\ -i(\bar{a}^2 - \bar{b}^2) & i(a\bar{b} + \bar{a}b) \end{pmatrix}$$

$$Fe_2F^{-1} = \begin{pmatrix} a\bar{b} - \bar{a}b & a^2 + b^2 \\ -(\bar{a}^2 + \bar{b}^2) & -(a\bar{b} - \bar{a}b) \end{pmatrix}.$$

Then the double cover $SU_2 \to SO_3$ is given by

$$F \mapsto \begin{pmatrix} a_0^2 + a_1^2 - b_0^2 - b_1^2 & -2(a_0b_0 + a_1b_1) & -2(a_0b_1 - a_1b_0) \\ 2(a_0b_0 - a_1b_1) & a_0^2 - a_1^2 - b_0^2 + b_1^2 & -2(a_0a_1 + b_0b_1) \\ 2(a_0b_1 + a_1b_0) & 2(a_0a_1 - b_0b_1) & a_0^2 - a_1^2 + b_0^2 - b_1^2 \end{pmatrix}. \tag{5.7}$$

Lemma 5.1.1. *Let $F : \mathbb{R} \to SU_2$ and $G : \mathbb{R} \to SO_3$ be differentiable mappings such that $F(t)$ and $G(t)$ correspond as in* (5.6) *for all t and which satisfy*

$$G' = GA \text{ and } F' = \tfrac{1}{2}F\xi \tag{5.8}$$

Then

$$A = \begin{pmatrix} 0 & -\xi_1 & \xi_2 \\ \xi_1 & 0 & -\xi_0 \\ -\xi_2 & \xi_0 & 0 \end{pmatrix} \text{ if and only if } \xi = \xi_0e_0 + \xi_1e_1 + \xi_2e_2. \tag{5.9}$$

Proof. The double cover is given by (5.6) Differentiating with respect to s yields

$$GAE_i \leftrightarrow \tfrac{1}{2}F[\xi, e_i]F^{-1},\ i = 0, 1, 2. \tag{5.10}$$

Hence

$$AE_i \leftrightarrow \tfrac{1}{2}[\xi, e_i]$$

which yields the matrix A. □

5.1.1 The double cover for SO_4

The action of SO_4 on S^3 is $X \mapsto GX$. The action of $SU_2 \times SU_2$ on SU_2 is $x \mapsto F_1xF_2^{-1}$.

Proposition 5.1.2. *Let $F_i : \mathbb{R} \to SU_2$, for $i = 1, 2$ and $G : \mathbb{R} \to SO_3$ be differentiable mappings such that $F_1(t)$, $F_2(t)$ and $G(t)$ satisfy*

(i) $G' = GA$ and $F_1' = \frac{1}{2}F_1\xi_1, F_2' = \frac{1}{2}F_2\xi_2$ for all t, with

$$\xi_1 = \lambda_1e_0 + xe_1 + ye_2 \text{ and } \xi_2 = -\lambda_1e_0 + xe_1 + ye_2,$$

(ii) $GE_k \leftrightarrow F_1e_kF_2^{-1}$ for $k = 0, 1, 2, 3$ where for $k = 0, 1$ and 2, e_k is the basis for $\mathfrak{su}_2$ and e_3 is the 2×2 identity matrix and E_k is the standard basis for $\mathbb{R}^4$.

Then

$$A = \begin{pmatrix} 0 & -\frac{1}{2}(\lambda_1 - \lambda_2) & 0 & 0 \\ \frac{1}{2}(\lambda_1 - \lambda_2) & 0 & -y & x \\ 0 & y & 0 & -\frac{1}{2}(\lambda_1 + \lambda_2) \\ 0 & -x & \frac{1}{2}(\lambda_1 + \lambda_2) & 0 \end{pmatrix}.$$

Proof. Differentiating the correspondence

$$GE_k \leftrightarrow F_1 e_k F_2^{-1}$$

yields

$$GAE_k \leftrightarrow \frac{1}{2}F_1\xi_1 e_k F_2^{-1} - \frac{1}{2}F_1 e_k \xi_2 F_2^{-1}$$
$$GAE_k \leftrightarrow \frac{1}{2}F_1(\xi_1 e_k - e_k\xi_2)F_2^{-1}$$

so

$$AE_k \leftrightarrow \frac{1}{2}(\xi_1 e_k - e_k\xi_2)$$

Then

(i) $\frac{1}{2}(\xi_1 e_0 - e_0\xi_2) = -\frac{1}{2}(\lambda_1 - \lambda_2)e + ye_1 - xe_2$.

(ii) $\frac{1}{2}(\xi_1 e_1 - e_1\xi_2) = -ye_0 + \frac{1}{2}(\lambda_1 + \lambda_1)e_2$.

(iii) $\frac{1}{2}(\xi_1 e_2 - e_2\xi_2) = xe_0 - \frac{1}{2}(\lambda_1 + \lambda_2)e_1$.

(iv) $\frac{1}{2}(\xi_1 e_3 - e_3\xi_2) = \frac{1}{2}(\lambda_1 - \lambda_2)e_3$.

□

5.2 The Parallel Frame

Given a Frenet frame F, a (non-unique) parallel frame G is defined by

$$G = F\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{5.11}$$

where $\theta = \int \tau(x)dx$. If F satisfies the Frenet-Serret differential equation

$$F' = F \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$$

then the parallel frame G satisfies the ODE

$$G' = G \begin{pmatrix} 0 & -\kappa\cos\theta & -\kappa\sin\theta \\ \kappa\cos\theta & 0 & 0 \\ \kappa\sin\theta & 0 & 0 \end{pmatrix}. \tag{5.12}$$

The SO_3 parallel frame equation (5.12), by lemma 5.1.1 can be rewritten in SU_2 as:

Lemma 5.2.1. *Let G satisfy* (5.12)*. Let F be a lift of G under the double cover $SU_2 \to SO_3$. Then F satisfies the differential equation*

$$F' = \frac{1}{2}F \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}. \tag{5.13}$$

where $q = \frac{1}{2}\kappa e^{i\int \tau(x)dx}$.

A curve γ with complex curvature q can then be computed by integrating

$$\gamma' = F e_o F^{-1}. \tag{5.14}$$

By Lemma 5.1.1, the Frenet frame equation in SO_3 can be rewritten in SU_2 as

$$F_1' = \frac{1}{2}F_1\ \alpha_1,\ \alpha_1 := \begin{pmatrix} i\tau & \kappa \\ -\kappa & -i\tau \end{pmatrix}.$$

Definition 5.2.2. The gauge action of $g : \mathbb{R} \to GL_2(\mathbb{C})$ on a map $\alpha : \mathbb{R} \mapsto \mathfrak{sl}_2(\mathbb{C})$ is

$$\alpha.g := g^{-1}\alpha g + g^{-1}g'. \tag{5.15}$$

Set

$$\alpha_2 = \alpha_1.g = \frac{1}{2}\begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$$

where $q = \kappa\exp(i\int \tau(x)dx)$. The parallel frame F_2 satisfying $F_2' = F_2\alpha_2$ is related to the Frenet frame F_1 by $F_2 = F_1 g$ where g is the diagonal gauge

$$g := \begin{pmatrix} \exp(-\frac{i}{2}\int\tau(x)dx) & 0 \\ 0 & \exp(-\frac{i}{2}\int\tau(x)dx) \end{pmatrix}.$$

Since α_2 is off diagonal, F_2 is a parallel frame. Now suppose γ is a closed, arclength parametrized curve with length L. While α_1 is periodic, g and α_2 are not. This can be repaired by considering the frame $F_3 := F_2.h$ satisfying $F_3' = F_3\alpha_3$ where h is the diagonal gauge

$$h := \begin{pmatrix} \exp(\frac{i}{2}\lambda s) & 0 \\ 0 & \exp(-\frac{i}{2}\lambda s) \end{pmatrix}$$

with $\lambda := \frac{1}{L}((\int\tau(x)dx) + 2\pi\mathbb{Z})$. Then $\alpha_3 = \alpha_2.h = \frac{1}{2}\begin{pmatrix} i\lambda & \widehat{q} \\ -\overline{\widehat{q}} & -i\lambda \end{pmatrix}$ where $\widehat{q} := q\exp(-i\lambda s) = \kappa\exp(-i\lambda s + i\int\tau(x)dx)$. Then α_3 is periodic by the choice of λ.

5.3 Sym formulas

We present a method of recovering a curve from its frame following Sym [30]. Let $F_\lambda : \mathbb{R} \to SU_2$ be a family of frames satisfying

$$F_\lambda^{-1}F_\lambda' = \frac{1}{2}\begin{pmatrix} i\lambda & q(t) \\ -\bar{q}(t) & -i\lambda \end{pmatrix} dt = \frac{1}{2}\xi dt. \tag{5.16}$$

We call ξ the potential for the curve. We can then generate the corresponding curves using the *Sym formulas* [30] given by

$$\mathbb{R}^3 : \gamma_\lambda := 2\left(\frac{d}{d\lambda}F_\lambda\right)F_\lambda^{-1},\ \lambda \in \mathbb{R}, \tag{5.17a}$$

$$\mathbb{S}^3 : \gamma_{\lambda_1,\lambda_2} := F_{\lambda_1}F_{\lambda_2}^{-1},\ \lambda_1,\ \lambda_2 \in \mathbb{R} \text{ distinct}, \tag{5.17b}$$

where λ in (5.17a) and λ_1, λ_2 in (5.17b) are called the sympoints.

Proposition 5.3.1. *The spacecurves γ in* (5.17a) *and* (5.17b) *are immersions into their respective spaceforms. Moreover*

(i) In $\mathbb{R}^3$, with sympoint λ_0, the curve γ is unit speed and complex curvature function $qe^{i\lambda_0 t}$.

(ii) In $\mathbb{S}^3$, with sympoints λ_1 and λ_2, the curve γ has arclength $\left(\frac{1}{2}(\lambda_1-\lambda_2)\right)t$ and complex curvature function $\left(\frac{1}{2}(\lambda_1-\lambda_2)\right)^{-1} qe^{\frac{i(\lambda_1+\lambda_2)t}{2}}$

Proof. The theorem is first proved in the following special cases:

(i) $\lambda = 0$ in the $\mathbb{R}^3$ sym formula (5.17a),

(ii) $\lambda_1 = 1$ and $\lambda_2 = -1$ in the $\mathbb{S}^3$ sym formula (5.17b).

Then the curve γ is arclength parameterized and q is its complex curvature. Firstly the proof of (ii), the other case is similar. Let $e_0 = \mathbb{1}$ and let e_1, e_2 and e_3 be the orthonormal basis for $\mathfrak{su}_2$ and let E_0, E_1, E_2, E_3 be the standard basis for $\mathbb{R}^4$. Consider the real vector space isomorphism $span_{\mathbb{R}}(e_0, e_1, e_2, e_3) \leftrightarrow \mathbb{R}^4$ defined by extending linearly the correspondence $e_k \leftrightarrow E_k, k = 0, 1, 2, 3$. Let $G : \mathbb{R} \to SO_4$ be the frame corresponding to (F_1, F_{-1}) under the double cover $SU_2 \times SU_2 \to SO_4$. Let $\xi := F^{-1}F'$ and $A := G^{-1}G'$. Differentiating the two sides of the correspondence

$$GE_k \leftrightarrow F_1 e_k F_{-1}^{-1}$$

yields the correspondence

$$AE_k \leftrightarrow \tfrac{1}{2}(\xi_1 e_k - e_k \xi_{-1})$$

and hence

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\mathrm{Im}(q) & \mathrm{Re}(q) \\ 0 & \mathrm{Im}(q) & 0 & 0 \\ 0 & -\mathrm{Re}(q) & 0 & 0 \end{pmatrix}.$$

It follows that G is a parallel frame for the curve

$$\gamma = F_1 e_0 F_{-1} \leftrightarrow FE_0$$

and that γ is arclength parametrized with complex curvature q. Similarly in (i)

$$AE_k \leftrightarrow Fe_k F^{-1}$$

which yields

$$\begin{pmatrix} 0 & -\mathrm{Im}(q) & \mathrm{Re}(q) \\ \mathrm{Im}(q) & 0 & -1 \\ -\mathrm{Re}(q) & 1 & 0 \end{pmatrix}$$

and G is then a parallel frame for the curve

$$\gamma = Fe_0F \leftrightarrow GE_0.$$

For the general case of arbitrary sympoints, the speed and complex curvature of γ can be computed by performing on F a coordinate change and a diagonal gauge of the form $g := diag(e^{im\lambda}, e^{-im\lambda})$ so that ξ and the evaluation points are as in the above proof, that is

$$\xi = \frac{1}{2}\begin{pmatrix} i\lambda & q \\ -\bar{q} & -i\lambda \end{pmatrix}$$

and in (i) $\lambda = 0$, (ii) $\lambda_1 = 1, \lambda_2 = -1$. Note that with
$g_1 := (exp(-mte_0/2)), m \in \mathbb{R}$

$$\xi.g_1 = \frac{1}{2}\begin{pmatrix} i(\lambda - m) & qe^{imt}) \\ -\bar{q}e^{-imt} & -i(\lambda - m) \end{pmatrix}.$$

For the $\mathbb{R}^3$ case with sympoint λ_0 and setting $m = \lambda_0$

$$\xi.g_1 = \frac{1}{2}\begin{pmatrix} i(\lambda - \lambda_0) & qe^{i\lambda_0 t} \\ -\bar{q}e^{-i\lambda_0 t} & -i(\lambda - \lambda_0) \end{pmatrix}.$$

By the first part of the proof, the spacecurve γ_{λ_0} has speed 1 and complex curvature $qe^{i\lambda_0 t}$ and by proposition 5.3.3 the curves induced by ξ and $\xi.g$ are the same up to isometry. For the S^3 case with $m = \frac{(\lambda_1+\lambda_2)}{2}$, after a coordinate change, $s = \frac{1}{2}(\lambda_1 - \lambda_2)t$, $W = \xi.g$ is

$$W|_{\lambda=\lambda_1} = \frac{1}{2}\begin{pmatrix} 1 & p \\ -\bar{p} & -1 \end{pmatrix}$$

$$W|_{\lambda=\lambda_2} = \frac{1}{2}\begin{pmatrix} -1 & p \\ -\bar{p} & 1 \end{pmatrix}$$

where

$$p = \frac{1}{\lambda_1 - \lambda_2} qe^{\dfrac{i(\lambda_1+\lambda_2)t}{2}}.$$

The expressions for speed and curvature then follow by step 1. □

To simplify calculations we will sometimes gauge the potential ξ of a curve. We show that under certain conditions this generates the same curve.

Lemma 5.3.2. *Let A satisfy $A^{'} = A\alpha$. Then $B := Ag$ satisfies $B^{'} = B\beta$ where*

$$\beta = g^{-1}\alpha g + g^{-1}g'. \tag{5.18}$$

Proof. $\beta = B^{-1}B' = g^{-1}A^{-1}(Ag)' = g^{-1}A^{-1}(A'g + Ag') = g^{-1}\alpha g + g^{-1}g'$. □

Proposition 5.3.3. *If $g : \mathbb{R} \to SU_2$ is a λ-independent diagonal guage, then*

(i) in $\mathbb{R}^3$ the spacecurves induced by ξ and $\xi.g$ are the same up to an ambient isometry.

(ii) in S^3 the curves induced by $\{\xi_1, \xi_2\}$ and $\{\xi_1.g, \xi_2.g\}$ are the same up to an ambient isometry.

Proof. (i) Let ξ be a potential and $F' = F\xi, F(0) = \mathbb{1}$. The spacecurve induced by F is $\gamma := 2\dot{F}F^{-1}$ where the dot denotes the derivative with respect to λ. With $g_0 := g(0)$, $\xi.g$ is a potential of the correct form, and the solution to $G' = G(\xi.g), G(0) = \mathbb{1}$, is $G = g_0^{-1}Fg$. The induced spacecurve is $\bar{\gamma} = g_0^{-1}\gamma g_0$ which is the same as γ up to the ambient isometry of $\mathbb{R}^3$ $x \mapsto g_0^{-1}xg_0$.

(ii) Let $\{\xi_1, \xi_2\}$ be potentials and $F'_{\lambda_i} = F_{\lambda_i}\xi_{\lambda_i}, F_{\lambda_i}(0) = \mathbb{1}$. The induced curve is $\gamma_{\lambda_1,\lambda_2} = F_{\lambda_1}F_{\lambda_2}^{-1}$. As before, let $g_0 := g(0)$, then the solution to $G'_{\lambda_i} = G_{\lambda_i}(\xi_{\lambda_i}.g), G_{\lambda_i}(0) = \mathbb{1}$ is $G_{\lambda_i} = g_o^{-1}F_{\lambda_i}g$ and the induced curve is $\overline{\gamma_{\lambda_1,\lambda_2}} = g_0^{-1}\gamma_{\lambda_1,\lambda_2}g_0$.

□

5.4 The nonlinear Schrödinger hierarchy

5.4.1 The Lax equation

We now turn to a study of the Lax equation associated with a frame which will allow us to develop the spectral theory of finite gap curves. A Lax equation is an equation of the form

$$X'(t_0) = [X(t_0), V(t_0)]. \tag{5.19}$$

where prime denotes the derivative with respect to the variable t_0.

If X satisfies

$$X' = [X, V],\ X(0) = \overset{o}{X} \tag{5.20}$$

and F satisfies

$$F' = FV,\ F(0) = \mathbb{1} \tag{5.21}$$

then F and X are related by

$$X = F^{-1}\overset{o}{X}F. \tag{5.22}$$

Proof. Let $Y := F^{-1}\overset{o}{X}F$. Differentiation shows that Y satisfies

$$Y' = [Y, V],\ Y(0) = \overset{o}{X}. \tag{5.23}$$

Since X also satisfies these equations, $Y = X$. □

Lemma 5.4.2. *For* $X \in \mathfrak{sl}_2(\mathbb{C})$*, the Lax equation* (5.19) *preserves the eigenvalues of X. In particular, it preserves det X.*

Proof. By the Cayley-Hamilton theorem

$$X^2 + \det(X)\mathbb{1} = 0. \tag{5.24}$$

Taking the trace

$$\det(X) = -\frac{1}{2}\mathrm{tr}(X^2). \tag{5.25}$$

Differentiation yields

$$(\det(X))' = -\frac{1}{2}d(\mathrm{tr}(X^2)) = \mathrm{tr}(X)X' = \mathrm{tr}(X)[X, V] = \mathrm{tr}(X^2V - XVX) = 0. \tag{5.26}$$

using the fact that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. □

Lemma 5.4.3. *If X satisfies X'=[X, V] then* $Y := g^{-1}Xg$ *satisfies* $Y' = [Y, V.g]$.

Proof.

$$\begin{aligned}[Y, V.g] &= [g^{-1}Xg, g^{-1}Vg + g^{-1}g'] \\ &= g^{-1}[X, V]g + (g^{-1})'Xg + g^{-1}Xg' \\ &= g^{-1}X'g + (g^{-1})'Xg + g^{-1}Xg' \\ &= (g^{-1}Xg)' \\ &= Y'.\end{aligned}$$

□

The NLS hierarchy can be seen as an infinite sequence of integrability conditions on a Lax equation

$$X' = \frac{1}{2}[X, V(X)] \tag{5.27}$$

on the formal power series in λ with coefficients in $\mathfrak{su}_2$

$$X = \sum_{k=0}^{\infty} X_k \lambda^k. \tag{5.28}$$

In the case of a polynomial Killing field $X = \sum_{k=0}^{d} X_k \lambda^k$ we have

$$V(X) := \sum_{c=0}^{d} V_{d,c}(X) dt_c \tag{5.29}$$

where

$$V_{d,c}(X) := (\lambda^{c-d} X)_+ \tag{5.30}$$

and where for $Y := \sum_{k=-\infty}^{\infty} Y_k \lambda^k$ we define $Y_+ = \sum_{k=0}^{\infty} Y_k \lambda^k$. The first few terms in $V(X)$ are

$$V_{d,0}(X) = X_d \tag{5.31a}$$

$$V_{d,1}(X) = X_{d-1} + \lambda X_d \tag{5.31b}$$

$$V_{d,2}(X) = \lambda^2 X_d + \lambda X_{d-1} + X_{d-2} \tag{5.31c}$$

$$V_{d,3}(X) = \lambda^3 X_d + \lambda^2 X_{d-1} + \lambda X_{d-2} + X_{d-3}. \tag{5.31d}$$

5.4.2 The NLS hierarchy

Let X be a polynomial Killing field. Then since $\det(X)$ is preserved by the Lax equation, its coefficients

$$a_k := \sum_{i+j=k} \langle X_i, X_j \rangle \tag{5.32}$$

are conserved quantities. Consider the Lax equation (5.19) along t_1. With prime denoting $\frac{d}{dt_1}$, expanding $X' = \frac{1}{2}[X, V_{d,1}(X)]$ in λ yields

$$X_0' + X_1'\lambda + \ldots + X_d'\lambda_d = \frac{1}{2}(X(X_{d-1} + \lambda X_d) - (X_{d-1} + \lambda X_d)X)$$

which leads to the recursive formula

$$X'_k = \frac{1}{2}[X_k, X_{d-1}] + \frac{1}{2}[X_{k-1}, X_d]. \tag{5.33}$$

The first two are

$$X'_d = 0 \tag{5.34}$$
$$X'_{d-1} = \frac{1}{2}[X_{d-2}, X_d]. \tag{5.35}$$

If X satisfies the Lax equation then its components can be computed recursively by the following two results.

Lemma 5.4.4. *For $X \in \mathfrak{su}_2$, $E \in \mathfrak{su}_2\backslash\{0\}$,*

$$X = \frac{1}{\langle E, E\rangle}(\langle X, E\rangle E + \tfrac{1}{2}[E, \tfrac{1}{2}[X, E]]). \tag{5.36}$$

Proof. Write $X = \begin{pmatrix} ix & -\bar\beta \\ \beta & -ix \end{pmatrix}$, $E = \begin{pmatrix} iy & -\bar\alpha \\ \alpha & -iy \end{pmatrix}$ Then

$$\frac{1}{\langle E, E\rangle} = \frac{2}{2y^2 + 2\alpha\bar\alpha}$$
$$\langle X, E\rangle E = \frac{1}{2}\begin{pmatrix} iy(2xy + \bar\alpha\beta + \alpha\bar\beta) & -\bar\alpha(2xy + \bar\alpha\beta + \alpha\bar\beta) \\ \alpha(2xy + \bar\alpha\beta + \alpha\bar\beta) & -iy(2xy + \bar\alpha\beta + \alpha\bar\beta) \end{pmatrix}.$$

$$\frac{1}{2}\left[E, \tfrac{1}{2}[X, E]\right] =$$
$$\frac{1}{2}\begin{pmatrix} i((2\alpha x - \beta y)\bar\alpha - \alpha y\bar\beta) & \bar\alpha^2\beta - 2y^2\bar\beta + \bar\alpha(2xy - \alpha\bar\beta) \\ 2y(-\alpha x + \beta y) + \alpha\bar\alpha\beta - \alpha^2\bar\beta & -i((2\alpha x - \beta y)\bar\alpha - \alpha y\bar\beta) \end{pmatrix}.$$

and this gives the result. □

Proposition 5.4.5. *Given $V_{d,1}(X) = X_{d-1} + \lambda X_d$, then X satisfying the Lax equation $X' = \frac{1}{2}[X, V_{d,c}(X)]$ can be computed recursively using Lemma 5.4.4 and*

$$\langle X_{d-k}, X_d\rangle = \tfrac{1}{2}(a_{2d-k} - \sum_{i=1}^{k-1}\langle X_{d-i}, X_{d-k+i}\rangle) = -\int\langle X'_{d-k+1}, X_{d-1}\rangle \tag{5.37a}$$
$$\tfrac{1}{2}[X_{d-k}, X_d] = X'_{d-k+1} - \tfrac{1}{2}[X_{d-k+1}, X_{d-1}]. \tag{5.37b}$$

Proof. Equation (5.37b) follows from (5.33). The first equality in (5.37a) follows from (5.32). To prove the last part of (5.37a), assume without loss of generality that $X_d = e_0$. Then

$$\begin{aligned}
\langle X_{d-k}, X_d\rangle' &= \langle X'_{d-k}, X_d\rangle + \langle X_{d-k}, X'_d\rangle \\
&= \langle X'_{d-k}, X_d\rangle \\
&= \langle \tfrac{1}{2}\left[X_{d-k}, X_{d-1}\right] + \tfrac{1}{2}\left[X_{d-k-1}, X_d\right], X_d\rangle \\
&= \langle \tfrac{1}{2}\left[X_{d-k}, X_{d-1}\right], X_d\rangle \\
&= -\tfrac{1}{2}\langle [X_{d-k}, X_d], X_{d-1}\rangle \\
&= -\langle X'_{d-k+1} - \tfrac{1}{2}\left[X_{d-k+1}, X_{d-1}\right], X_{d-1}\rangle \\
&= -\langle X'_{d-k+1}, X_{d-1}\rangle.
\end{aligned}$$

□

Writing prime for $\frac{d}{dt_0}$ and dot for $\frac{d}{dt_n}$, the integrability condition

$$\dot{V}_{d,1} - V'_{d,n} = \tfrac{1}{2}[V_{d,1}, V_{d,n}]$$

implies

$$\dot{X}_{d-1} = \tfrac{1}{2}[X_{d-n-1}, X_d].$$

We provide an explicit characterization of the NLS hierarchy and its mKdV subhierarchy.

Proposition 5.4.6. *(i) The recursion operator for the NLS hierarchy is*

$$\phi_{NLS}(f(t_0)) = if'(t_0) + \tfrac{i}{2}q(t_0)\int(\bar{q}(t_0)f(t_0) - q(t_0)\bar{f}(t_0)).$$

(ii) The mKdV hierarchy is the subhierarchy of even flows with $\kappa := q$ real-valued. Its recursion operator is

$$\phi_{mKdV}(f(t_0)) = -f''(t_0) - \kappa^2(t_0)f(t_0) - \kappa'(t_0)\int(\kappa(t_0)f(t_0)).$$

Proof. (i). Let $X_d = e_0$, $X_k = a_k e_0 + b_k e_1 + c_k e_2$ and $q_k = b_k + ic_k$. From (5.37b) we obtain

$$a'_{k-1} = \frac{i}{2}(\overline{q_{d-1}}q_{d-k+1} - q_{d-1}\overline{q_{d-k+1}}) \tag{5.38}$$

$$q_{d-k} = iq_{d-k+1} + a_{d-k+1}q_{d-1} - a_{d-1}q_{d-k+1}. \tag{5.39}$$

Hence

$$q_{d-k} = iq'_{d-k+1} - a_{d-1}q_{d-k+1} + \frac{i}{2}q_{d-1}\int(\overline{q_{d-1}}q_{d-k+1} - q_{d-1}\overline{q_{d-k+1}}) \tag{5.40}$$

The choice $a_{d-1} = 0$ yields the NLS recursion operator ϕ_{NLS}. (ii). The mKdV hierarchy with $a_{d-1} = 0$, q_{2k+1} real and q_{2k} imaginary is obtained by applying the NLS recursion operator ϕ_{NLS} twice. Assume without loss of generality that d is odd (a similar calculation follows for d even). Then q_{d-1} and q_{d-2k} are imaginary, then $\overline{q_{d-1}}q_{d-2k} - q_{d-1}\overline{q_{d-2k}} = 2q_{d-1}q_{d-2k}$. Thus

$$q_{d-2k-1} = -q'_{d-2k} + i\int q_{d-1}q_{d-2k}. \tag{5.41}$$

Since q_d and q_{d-2k+1} are real, $\overline{q_{d-1}}q_{d-2k+1} - q_{d-1}\overline{q_{d-2k+1}} = 0$. Thus

$$q_{d-2k-2} = iq'_{d-2k-1}.$$

Hence

$$\begin{aligned} q_{d-2k-2} &= -q''_{d-2k+1} - \left(q_{d-1}\int q_{d-1}q_{d-2k+1}\right) \\ &= -q''_{d-2k+1} - q^2_{d-1}q_{d-2k+1} - q'_{d-1}\int q_{d-1}q_{d-2k+1}. \end{aligned}$$

This yields the mKdV recursion operator ϕ_{mKdV}. □

Let dot denote the derivative with respect to the flow parameter. Then flows 0,1 and 2 in the nonlinear Schrödinger hierarchy are

$$i\dot{q} = q \tag{5.42a}$$
$$\dot{q} = q' \tag{5.42b}$$
$$-i\dot{q} = q'' + \tfrac{1}{2}|q|^2 q + (\tfrac{1}{8}a_1^2 - \tfrac{1}{2}a_2)q + \tfrac{i}{2}a_1 q'. \tag{5.42c}$$

While flows 0,1 and 2 in the mKdV hierarchy are

$$\dot{\kappa} = -\kappa' \tag{5.43a}$$
$$\dot{\kappa} = -\kappa''' - \frac{3}{2}\kappa^2\kappa' \tag{5.43b}$$
$$\dot{\kappa} = \kappa''''' - \frac{15}{8}\kappa^4\kappa' - \frac{5}{2}\kappa^2\kappa''' - 10\kappa\kappa'\kappa''. \tag{5.43c}$$

The first few terms in X are

$$X_d = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$X_{d-1} = \begin{pmatrix} \frac{i}{2}a_{2d-1} & q \\ -\bar{q} & -\frac{i}{2}a_{2d-1} \end{pmatrix}$$

$$X_{d-2} = \begin{pmatrix} -\frac{i}{2}|q|^2 - \frac{i}{8}a_{2d-1}^2 + \frac{i}{2}a_{2d-2} & iq' \\ i\bar{q}' & \frac{i}{2}|q|^2 + \frac{i}{8}a_{2d-1}^2 - \frac{i}{2}a_{2d-2} \end{pmatrix}.$$

The frame associated to a polynomial Killing field X is the solution to

$$dF = \tfrac{1}{2}FV(X),\ F(0) = \mathbb{1}. \tag{5.45}$$

With the preceding we are now ready to prove

Lemma 5.4.7. *Let $\Lambda_d\mathfrak{su}_2$ be the set of all degree d polynomials with coefficients in $\mathfrak{su}_2$. For any 2d points $p_1, \bar{p}_1, \dots, p_d, \bar{p}_d \in \mathbb{C}$, there exists $\overset{o}{X} \in \Lambda_d\mathfrak{su}_2$ with leading coefficient e_0 satisfying*

$$\det \overset{o}{X} = \prod_{k=1}^{d}(\lambda - p_k)(\lambda - \bar{p}_k).$$

Moreover, $\overset{o}{X}$ may be chosen to lie in span(e_0, e_1).

Proof. For any meromorphic function $f : \mathbb{C} \to \mathbb{C}$, define $f^*(\lambda) = \overline{f(\bar{\lambda})}$. With $g(\lambda) := \prod_{k=1}^{d}(\lambda - p_k)$, let

$$\overset{o}{X} := \tfrac{1}{2}(g + g^*)e_0 + \tfrac{1}{2i}(g - g^*)e_1.$$

Then $\overset{o}{X} \in \Lambda_d\mathfrak{su}_2$, $\overset{o}{X}$ lies in the span of e_0 and e_1 and the coefficient of λ^d is e_0. Compute

$$\begin{aligned} \det \overset{o}{X} &= \tfrac{1}{4}(g + g^*)^2 - \tfrac{1}{4}(g - g^*)^2 \\ &= gg^* \\ &= \prod_{k=1}^{d}(\lambda - p_k)(\lambda - \bar{p}_k) \end{aligned}$$

so $\overset{o}{X}$ has the required determinant. $\square$

5.5 The spectral curve

We now turn to developing the theory of the spectral curve associated with a finite gap curve. Let $SL_2(\mathbb{C})$ be the special linear group of 2×2 complex matrices with determinant 1. Its Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ consists of 2×2 complex matrices with trace 0 and has a basis consisting of

$$f_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, f_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\overset{o}{X} : \mathbb{C} \to sl_2(\mathbb{C})$ be a polynomial with $\det(X_0) \neq 0$ and assume that all the zeros of $\det(\overset{o}{X})$ are simple. The spectral curve associated to $\overset{o}{X}$ can be defined starting with the set

$$\Sigma_0 = \left\{ (\lambda, \mu) \mid \mu = \sqrt{-\det \overset{o}{X}(\lambda)} \right\} \tag{5.46}$$

Compactifying Σ_0 by adding two points at ∞ yields a smooth hyperelliptic Riemann surface Σ that is branched at p_k and $\bar{p_k}$. It is called the spectral curve associated to X.

Proposition 5.5.1. *The genus of Σ is $g = (deg\overset{o}{X}) - 1$.*

Proof. Define $\pi : \Sigma \to \mathbb{C}P^1$ by $(\mu) \to \lambda$. Then π is a degree 2 branched cover of Σ with ramification index 2 at each branch point. Then, denoting the genus of Σ by g_Σ and the degree of X by g, by the Riemann-Hurwitz formula

$$\begin{aligned} 2g_\Sigma - 2 &= 2(2\ \text{genus}(\mathbb{C}P^1) - 2) + \sum_{\text{p a branch point}} (e_p - 1) \\ 2g_\Sigma - 2 &= -4 + 2g \\ g_\Sigma &= g - 1. \end{aligned}$$

□

By Lemma 5.4.1, $X(t) = F^{-1}\overset{\circ}{X}F$ and the spectral curves associated to $X(t)$ and $\overset{o}{X} = X(0)$ are the same.

Example 5.5.1.

(i) The Vacuum:

The situation in which the degree of the polynomial killing field is 0 is the vacuum. Let $X = e_0$, then $V_{0,1}(X) = \lambda e_0$. The Lax equation is then

$$X' = \frac{1}{2}[X, \lambda e_0],\ X(0) = e_0$$

the solution of which is the constant $X = e_0$. The frame F satisfying

$$F' = \tfrac{1}{2}F\lambda e_0,\ F(0) = \mathbb{1}$$

is

$$F = exp(\tfrac{1}{2}\lambda t e_0) = \begin{pmatrix} e^{\frac{i\lambda t}{2}} & 0 \\ 0 & e^{-\frac{i\lambda t}{2}} \end{pmatrix}. \tag{5.47}$$

Then the spacecurve generated by the Sym formulas (5.17a) and (5.17b) are given by

(a) In $\mathbb{R}^3$, $\gamma_{\mathbb{R}^3} = te_0$ which is a straight line,

(b) In S^3, $\gamma_{S^3} = exp(\frac{1}{2}it(\lambda_1 - \lambda_2)e_0)$, which is a great circle.

(ii) Spectral Genus 0:
For spectral genus 0, the degree of the polynomial killing field X is 1 and so $X = e_0 + \lambda X_0$ and $V_{1,1} = X$ and so the Lax equation is $X' = \frac{1}{2}[X, V_{1,1}(X)]$ and hence X is constant. The frame F satisfying $F' = \frac{1}{2}FV_{1,1}(X)$ is

$$F = exp(\tfrac{1}{2}tX). \tag{5.48}$$

With sympoint $\lambda_{sym} = 0$, the spacecurve in $\mathbb{R}^3$ is a circle if the zeros of $\det(X_1)$ are in $i\mathbb{R}$ and a helix otherwise.

Definition 5.5.2. Let τ be a shift map on the line and X a polynomial Killing field satisfying $X \circ \tau = X$ and F a frame associated to X, that is F is a solution to $F' = \frac{1}{2}FV(X)$ and let τ be a translation of $\mathbb{R}$. Then $F \circ \tau$ is also a solution and $F \circ \tau = MF$ for some matrix M. We call M the monodromy matrix with respect to τ and it is given by $M := (\tau^* F)F^{-1}$.

5.6 Closing conditions

In order to generate flat tori from finite gap curves we require closure of the admissible curves on $\mathbb{S}^2$ and as such the closing conditions are necessary and sufficient conditions that a spacecurve is closed, they consist of:

- The intrinsic closing condition: the complex curvature is periodic.

- The extrinsic closing conditions: conditions on the monodromy at the sympoint.

We will say that a frame F is monodromic if its potential $F^{-1}F'$ is periodic. This is equivalent to the intrinsic closing condition. Suppose that γ is periodic so that $\tau^*\gamma = \gamma$ for some translation $\tau(s) = s + p, p \in \mathbb{R}, p \neq 0$. Then the curvature κ and torsion τ of γ are periodic. Geometrically, F has monodromy which may be interpreted as a rotation in the axis along $\gamma'(0)$. In this case a potential for F, ξ can be gauged by an appropriate diagonal gauge $g = exp(\frac{1}{2}ipe_0)$ so that $\xi.g$ is periodic and induces the same curve as ξ. The gauged frame F then has monodromy $\mathbb{1}$. To describe the extrinsic closing conditions, let $\tau(s) = s + p$ be a translation of $\mathbb{R}$ by $p \in \mathbb{R}$. Suppose $\tau^*(V_{d,1}) = V_{d,1}$ which is the intrinsic closing condition. Let $dF = \frac{1}{2}FV$ and let $M = (\tau^*F)F^{-1}$ be the monodromy of F.

Proposition 5.6.1 (Extrinsic closing condition 1)**.** *[19] If a frame F is monodromic with monodromy M, then if γ, the induced curve in $\mathbb{R}^3$, contains three linearly independent points, then γ is closed if and only if $M|_{\lambda_0} \in \{\pm\mathbb{1}\}$ and $\frac{d}{d\lambda}M|_{\lambda_0} = 0$. where λ_0 is the sympoint.*

Proof. Let dot denote the derivative with respect to λ. For $\mathbb{R}^3$ the sym formula is $\gamma = 2\frac{dF}{d\lambda}F^{-1}$. Let τ be the translation of the arclength parameter so that $\tau^*F = MF$. Then $\tau^*\gamma = M\gamma M^{-1} + 2\dot{M}M^{-1}$ since

$$\begin{aligned}\tau^*\gamma &= \tau^*(2\dot{F}F^{-1}) \\ &= 2\tau^*\dot{F}\tau^*F^{-1} \\ &= 2\tau^*\dot{F}F^{-1}M^{-1}.\end{aligned}$$

and differentiating $\tau^*F = MF$ yields $(\tau^*\dot{F}) = \dot{M}F + M\dot{F}$ and thus $\tau^*F = \dot{M}F + M\dot{F}$. Hence

$$\begin{aligned}\tau^*\gamma &= 2\dot{M}FF^{-1}M^{-1} + 2M\dot{F}F^{-1}M^{-1} \\ &= 2\dot{M}M^{-1} + M\gamma M^{-1}.\end{aligned}$$

If the closing conditions for $\mathbb{R}^3$ are satisfied then γ is closed. Conversely, since γ is non planar, there exists three points on γ which span $\mathbb{R}^3$ and thus we have an isometry that fixes a basis of $\mathbb{R}^3$ and thus must be $\pm\mathbb{1}$ and it follows that $M = \pm\mathbb{1}$ and $\dot{M} = 0$. $\square$

Since $M \in SU_2$ we also have the following result.

Proposition 5.6.2 (Extrinsic closing conditions 2)**.** *If a frame F is monodromic with monodromy eigenvalue μ, the induced spacecurve γ in $\mathbb{R}^3$ is closed if and only if*

$$\mu|_{\lambda_0} \in \{\pm\mathbb{1}\} \text{ and } \frac{d}{d\lambda}\mu|_{\lambda_1} = 0.$$

5.7 Isospectral deformations and monodromy

Proposition 5.7.1. *X is periodic if $V_{d,1}(X)$ is periodic.*

Proof. The recursive formulas (5.37a) and (5.37b) define periodic X_k when $X_{k+1}, \dots, X_d$ are periodic. □

Proposition 5.7.2. *Under isospectral flows, a monodromic frame remains monodromic and its monodromy is preserved. In particular, an isospectral deformation of a closed curve is closed.*

Proof. On the (t_1, t_2) plane let X satisfy

$$dX = \frac{1}{2}[X, V_{d,1}(X)dt_1 + V_{d,2}(X)dt_2]$$

and F satisfy

$$F' = \frac{1}{2}F(V_{d,1}(X)dt_1 + V_{d,2}(X)dt_2).$$

Let $\tau(t_1, t_2) = (t_1 + L, t_2)$, $L \in \mathbb{R}$. Suppose that $\tau^*V_{d,1}(X) = V_{d,1}(X)$ on $\{t_2 = 0\}$. By proposition 5.7.1, X satisfies

$$\tau^*X = X \text{ on } \{t_2 = 0\}.$$

Since τ^*X and X satisfy the same equation

$$Y' = \frac{1}{2}[Y, V_{d,2}(Y)dt_2]$$

with the same initial condition at $t_2 = 0$ we have $\tau^*X = X$ for all t_2. Let $M = (\tau^*F)F^{-1}$ be the monodromy of F. Then

$$\frac{d}{dt_2}M = (\tau^*F)(\tau^*V_{d,2}(X) - V_{d,2}(X))F^{-1} = 0.$$

Hence M is independent of t_2, that is M is invariant under isospectral deformation. Thus if the curve along $\{t_2 = 0\}$ satisfies the extrinsic closing conditions, then the isospectrally deformed curve also satisfies these closing conditions. $\square$

5.8 Spherical Curves

That a curve lies on a sphere can be characterized by its complex curvature by:
Proposition 5.8.1. *In the $\mathbb{R}^3$ theory, a spacecurve lies in a sphere if and only if the image of its complex curvature q lies in a straight line in $\mathbb{C}$.*

Proof. Let γ be a spacecurve with complex curvature q. Let F be the parallel frame of γ so $\gamma' = Fe_0F^{-1}$ and $\xi := F^{-1}F' \in span(e_1, e_2)$. Write $\gamma = FAF^{-1}$, where $A : \mathbb{R} \mapsto \mathfrak{su}_2$. Then $\gamma' = Fe_0F^{-1} = F(A' + [\xi, A])F^{-1}$. So

$$A' = e_0 - [\xi, A]. \tag{5.49}$$

Assume that γ lies in a sphere of radius $r \neq \infty$ centered at the origin. Then $r^2 = -\frac{1}{2}tr\gamma^2 = -\frac{1}{2}trA^2$. Hence $trAA' = 0$. By (5.49), $A \in span(e_1, e_2)$, so by (5.49) again $A' = 0$. Hence $[\xi, A] = e_0$. That is, let

$$A = \begin{pmatrix} ix & \beta \\ -\bar{\beta} & -ix \end{pmatrix}, \ \xi = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}.$$

Then

$$A' = e_0 - [\xi, A] = \begin{pmatrix} i + \bar{\beta}q - \beta\bar{q} & 2ixq \\ 2i\bar{q}x & -i - \bar{\beta}q + \beta\bar{q} \end{pmatrix}.$$

Thus $\mathrm{tr}AA' = 0$ implies that $x = 0$ and thus $A \in span(e_1, e_2)$. Then

$$A = \begin{pmatrix} 0 & \beta \\ -\bar{\beta} & 0 \end{pmatrix}$$

and hence

$$A' = \begin{pmatrix} 0 & \beta' \\ -\bar{\beta}' & 0 \end{pmatrix}$$

which implies that $A' = 0$. Hence $[\xi, A] = e_0$. This implies that the image of q lies in a straight line in $\mathbb{C}$ with distance $\frac{1}{r}$. Conversely, suppose that q lies in a straight line with distance $\frac{1}{r}$ to the origin. Then $[\xi, A_0] = e_0$ for some $A_0 \in \mathfrak{su}_2$ satisfying $-\frac{1}{2}trA_0 = r^2$. By a translation it may be assumed that $A(0) = A_0$.

Then $A(s) = A_0$ is the unique solution to the differential equation (5.49). It follows that $trAe_0 = 0$, so $trAA' = 0$. Hence $-\frac{1}{2}tr\gamma^2 = -\frac{1}{2}trA^2 = r^2$ and so γ lies on a sphere of radius r^2 centered at the origin. □

We now introduce the notion of twisted and antitwisted for a polynomial Killing field which will allow us to characterize spherical curves in terms of their polynomial Killing fields.

Definition 5.8.2. Let $L \subset \mathfrak{su}_2$ be a line through the origin. Then for a formal power series $X = \sum_{k=0}^{\infty} X_k\lambda^k$, we say that X is twisted with respect to L if

$$X_{2k} \in L \text{ and } X_{2k+1} \in L^{\perp},$$

and antitwisted if

$$X_{2k} \in L^{\perp} \text{ and } X_{2k+1} \in L.$$

Theorem 5.8.3. *In the $\mathbb{R}^3$ theory let γ be a finite gap curve and $X = \sum_{k=1}^{d} X_k\lambda_k$ its corresponding polynomial Killing field. Then if*

(i) d is odd and X is twisted with respect to some line, then γ is spherical.

(ii) d is even and X is antitwisted with respect to some line, then γ is spherical.

Proof. (i) Assume that d is odd and X is twisted with respect to some line L. Then since

$$X_{d-1} = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$$

we have that $X_{d-1} \in L$, that is the image of q lies in a line. Thus by proposition 5.8.1, γ is spherical. Similarly (ii) follows. □

5.9 Deformations

In this section we describe flows of spectral curves that will preserve the intrinsic and extrinsic closing conditions. Let Σ be the spectral curve arising from a polynomial Killing field X, $p_1, \ldots, p_{2g+2}$ its branchpoints and ∞_1 and ∞_2 the two points over $\lambda = \infty$. If z is a local coordinate on Σ near a branch point q, then locally $\lambda - q = z^2$ and so λ has a double zero at each branchpoint. Thus $d\lambda$ has a simple zero at each branchpoint. Then since $d\lambda$ is invariant under the hyperelliptic involution that switches sheets, the divisor of $d\lambda$ is

$$\mathrm{Div}(d\lambda) = \sum_{k=1}^{2g+2} [p_k] - 2[\infty_1] - 2[\infty_2].$$

Set

$$\eta^2 = \prod_{k=1}^{2g+2} (\lambda - p_k).$$

η then has simple zeroes at each branch point and no other finite zeros. The divisor of η is

$$\mathrm{Div}(\eta) = \sum_{k=1}^{2g+2} [p_k] - (g+1)[\infty_1] - (g+1)[\infty_2].$$

Definition 5.9.1. Let $\lambda \in \mathbb{C}$ and let ω_1 and ω_2 be the two points on the spectral curve Σ over λ and let $\mu_1(\lambda)$ and $\mu_2(\lambda)$ be the eigenvalues of the monodromy associated to the frame F with period 2π at λ. Then

(i) The common eigenvectors ϕ_i of the monodromy and the differential operator

$$\frac{d}{dx} - \frac{1}{2}\xi$$

where ξ is given by

$$\begin{pmatrix} i\lambda & q \\ -\bar{q} & -i\lambda \end{pmatrix}$$

with corresponding eigenvalues μ_i and 0 respectively, are called Bloch functions.

(ii) λ is called a regular point if $\mu_1 \neq \mu_2$, otherwise we say that λ is irregular.

(iii) An irregular point λ is called a double point if the monodromy around λ is trivial. It is called removable if the two Bloch functions $\phi_1(x, \lambda)$ and $\phi_2(x, \lambda)$ associated to μ_1 and μ_2 are not equal.

Let μ be an eigenvalue of the monodromy. We will express the deformation in terms of the λ and t derivatives of the log of μ. Firstly we derive expressions for $\mathrm{dlog}(\mu)$ and $\frac{\mathrm{d}}{\mathrm{dt}}\log(\mu)$ where t is a deformation parameter.

Lemma 5.9.2. *[18] Let $\lambda \in \mathbb{C}$ be a regular or removable double point, ω_1 and ω_2 be the two points on the spectral curve Σ over λ and μ an eigenvalue of the* *monodromy at λ. Then*

$$\mathrm{dlog}(\mu) = -\frac{1}{2i}\frac{\int_0^{2\pi}(\phi_1(x,\omega_1)\phi_2(x,\omega_2)+\phi_2(x,\omega_1)\phi(x,\omega_2))dx}{(\phi_1(0,\omega_1)\phi(0,\omega_2)-\phi_2(0,\omega_1)\phi_1(0,\omega_2))}d\lambda \tag{5.50}$$

where ϕ_1 and ϕ_2 are the Bloch functions at λ.

Proof. Let α and β be points in Σ and λ_α and λ_β their projections to the λ-plane. Then

$$\begin{aligned}&\frac{d}{dx}(\phi_1(x,\alpha)\phi_2(x,\beta)-\phi_2(x,\alpha)\phi_1(x,\beta))=\\&\phi_1'(x,\alpha)\phi_2(x,\beta)+\phi_1(x,\alpha)\phi_2'(x,\beta)-\phi_2'(x,\alpha)\phi_1'(x,\beta).\end{aligned} \tag{5.51}$$

Then since $\phi_i' = \frac{1}{2}\phi_i\xi$, we have

$$\begin{aligned}\phi_1'(x,y) &= \frac{1}{2}(i\lambda_y\phi_1(x,y)-\bar{q}\phi_2(x,y))\\ \phi_2'(x,y) &= \frac{1}{2}(q\phi_1(x,y)-i\lambda_y\phi_2(x,y)).\end{aligned}$$

Thus (5.51) is equivalent to

$$\frac{i}{2}(\lambda_\alpha-\lambda_\beta)(\phi_1(x,\alpha)\phi_2(x,\beta)+\phi_1(x,\beta)\phi_2(x,\alpha)) \tag{5.52}$$

Integrating between 0 and 2π we get

$$\begin{aligned}&\phi_1(2\pi,\alpha)\phi_2(2\pi,\beta)-\phi_2(2\pi,\alpha)\phi_1(2\pi,\beta)-\phi_1(0,\alpha)\phi_2(0,\beta)+\phi_2(0,\alpha)\phi_1(0,\beta)\\ =&\frac{i}{2}(\lambda_\alpha-\lambda_\beta)\int_0^{2\pi}\phi_1(x,\alpha)\phi_2(x,\beta)+\phi_1(x,\beta)\phi_2(x,\alpha)dx\end{aligned}$$

Define a function ζ on Σ by if $\lambda\in\mathbb{C}$, ω_1 and ω_2 the points on Σ over λ and $\mu_1(\lambda)$ and $\mu_2(\lambda)$ the eigenvalues of the monodromy at λ, then $\zeta(\omega_i)=\mu_i$. Since ϕ_i is an eigenvector for the monodromy

$$\phi_i(x+2\pi,y)=\zeta(y)\phi_i(x).$$

Thus (5.52) can be rewritten as

$$(\zeta(\alpha)\zeta(\beta)-1)(\phi_1(0,\alpha)\phi_2(0,\beta)-\phi_2(0,\alpha)\phi_1(0,\beta)) =$$
$$-2i(\lambda_\alpha-\lambda_\beta)\int_0^{2\pi}\phi_1(x,\alpha)\phi_2(x,\beta)+\phi_1(x,\beta)\phi_2(x,\alpha)dx.$$

Then since λ is a regular or removable double point

$$\phi_1(0,\alpha)\phi_2(0,\beta)-\phi_2(0,\alpha)\phi_1(0,\beta)\neq 0,$$

Thus (5.52) is equivalent to

$$\frac{\int_0^{2\pi}\phi_1(x,\alpha)\phi_2(x,\beta)+\phi_1(x,\beta)\phi_2(x,\alpha)dx}{\phi_1(0,\alpha)\phi_2(0,\beta)-\phi_2(0,\alpha)\phi_1(0,\beta)} = \frac{2\zeta(\alpha)\zeta(\beta)-1}{i(\lambda_\alpha-\lambda_\beta)}. \tag{5.53}$$

Now let $\alpha=\omega_1+\delta$ and $\beta=\omega_2$. Define a function f by $f(\omega)=\zeta(\omega)\zeta(\omega_2)$. Then taking the limit as $\delta\to 0$ the right hand side of (5.53) is

$$-2if'(\omega_1) = -2i\zeta'(\omega_1)\zeta(\omega_2) = -2i\frac{\zeta'(\omega_1)}{\zeta(\omega_1)} = -2i\mathrm{dlog}(\mu_1).$$

□

With this we can derive an expression for $\mathrm{dlog}(\mu)$ following Grinevich-Schmidt [18], [20]:

Corollary 5.9.3. $\mathrm{dlog}(\mu)=\dfrac{b(\lambda)}{\eta}d\lambda$ *where b is a monic polynomial of degree $g+1$.*

Proof. An asymptotic calculation shows that $\mathrm{dlog}(\mu)$ has a pole of order two at ∞ with no residues. It is holomorphic on the finite part of Σ and thus its divisor is of the form

$$Div(\mathrm{dlog}(\mu)) = \sum_{k=1}^{2g+2}[B_k]-2[\infty_1]-2[\infty_2].$$

The result then follows. □

Lemma 5.9.4. *If $\mathrm{dlog}(\mu)$ deforms in t so that the intrinsic period is preserved (that is, q is periodic), then $\frac{\mathrm{d}}{\mathrm{dt}}\log(\mu)$ is a meromorphic differential on Σ and is of the form*

$$\frac{\mathrm{d}}{\mathrm{dt}}\log(\mu)=\frac{c(\lambda)}{\eta}$$

for some polynomial $c(\lambda)$ of degree g.

Proof. $\log(\mu)$ is locally meromorphic on Σ with simple poles at ∞_1 and ∞_2. Moreover it is multivalued and consists of additive constants in $2\pi i\mathbb{Z}$. Thus $\frac{\mathrm{d}}{\mathrm{dt}}\log(\mu)$ is single valued and meromorphic on Σ. Since $\log(\mu)$ is locally like $\sqrt{\lambda - p(t)}$ at a branch point p, $\frac{\mathrm{d}}{\mathrm{dt}}\log(\mu)$ is locally like

$$\frac{1}{2}p'(t)(\lambda - p(t))^{-\frac{1}{2}},$$

thus $\frac{\mathrm{d}}{\mathrm{dt}}\log(\mu)$ has a simple pole at each branch point. An asymptotic calculation shows that $\frac{\mathrm{d}}{\mathrm{dt}}\log(\mu)$ has a simple zero at each of ∞_1, ∞_2. Thus the divisor of $\frac{\mathrm{d}}{\mathrm{dt}}\log(\mu)$ is

$$\mathrm{Div}(\frac{\mathrm{d}}{\mathrm{dt}}\log(\mu)) = -\sum_{k=1}^{2g+2}[p_k] + \sum_{k=1}^{2g}[C_k] + [\infty_1] + [\infty_2],$$

for some $C_1, \dots, C_{2g} \in \Sigma$. Thus $\frac{\mathrm{d}}{\mathrm{dt}}\log(\mu)$ can be written in the required form. □

Following this we have that the Whitham deformation which is the compatibility condition

$$\frac{\mathrm{d}}{\mathrm{dt}}(\mathrm{d}\log(\mu)) = \mathrm{d}(\frac{\mathrm{d}}{\mathrm{dt}}\log\mu),$$

in terms of the expressions for $\mathrm{d}\log(\mu)$ and $\frac{\mathrm{d}}{\mathrm{dt}}\log(\mu)$ this is equivalent to

$$\nu' b - b'\nu = \dot{\nu}c - \dot{c}\nu$$

where prime denotes the t derivative and dot the λ derivative. We require an additional condition to maintain the extrinsic closing condition.

Proposition 5.9.5. *For curves in $\mathbb{R}^3$, the extrinsic closing conditions are preserved by the flow at the sym point λ_{sym} if and only if*

$$\frac{\mathrm{d}}{\mathrm{dt}}\lambda_{sym} = -\frac{\frac{\mathrm{d}}{\mathrm{dt}}\log(\mu)}{\mathrm{d}\log(\mu)}\Big|_{\lambda_{sym}} \; \textit{and} \; (\mathrm{d}\log(\mu))|_{\lambda_{sym}} = 0. \tag{5.54}$$

If the sym point is t-independent then the extrinsic closing conditions are preserved by the deformation if and only if

$$(\frac{\mathrm{d}}{\mathrm{dt}}\log(\mu)|_{\lambda_{sym}}) = 0 \; \textit{and} \; (\mathrm{d}\log(\mu)|_{\lambda_{sym}}) = 0. \tag{5.55}$$

Proof. Recall that for curves in $\mathbb{R}^3$, by proposition 5.6.2 the extrinsic closing conditions were the requirements that

(i) $\mu|_{\lambda_{sym}} \in \{\pm 1\}$,

(ii) $\frac{d}{d\lambda}\mu|_{\lambda_{sym}} = 0$.

(i) is then equivalent to

$$\log(\mu)|_{\lambda_{sym}} \in i\pi\mathbb{Z}.$$

If $\log(\mu)$ depends on a deformation parameter t, this condition is preserved if

$$\frac{\mathrm{d}}{\mathrm{dt}}(\log(\mu)(\lambda_{sym}(t), t)) = 0.$$

Differentiating we then have

$$\frac{\mathrm{d}}{\mathrm{dt}}(\log(\mu)(\lambda_{sym}(t), t)) = (\mathrm{d}_\lambda \log(\mu))|_{(\lambda_{sym}(t),t)} \cdot \frac{\mathrm{d}}{\mathrm{dt}}\lambda_{sym} + (\frac{\mathrm{d}}{\mathrm{dt}}\log(\mu))|_{(\lambda_{sym}(t),t)}.$$

Hence the condition to satisfy (i) is

$$\lambda' = -\frac{\frac{\mathrm{d}}{\mathrm{dt}}\log(\mu)}{\mathrm{d}\log(\mu)}\Big|_{\lambda_{sym}}.$$

Furthermore, the condition to satisfy (ii) is then

$$(\mathrm{d}\log(\mu))|_{\lambda_{sym}} = 0.$$

If λ_{sym} then satisfies $\frac{\mathrm{d}}{\mathrm{dt}}\lambda_{\mathrm{sym}} = 0$, then $b(\lambda)$ has a zero at λ_{sym} since $(\mathrm{d}\log(\mu)) = 0$. Thus $c(\lambda)$ has a double zero at λ_{sym} since $\frac{\mathrm{d}}{\mathrm{dt}}(\lambda_{sym}) = 0$. Thus (5.54) and (5.55) are equivalent assuming that $\frac{\mathrm{d}}{\mathrm{dt}}\lambda_{sym} = 0$. □

Chapter 6

Finite Type Flat Surfaces in $\mathbb{S}^3$

In this chapter we prove that a curve on $\mathbb{S}^2$ is determined wholly by its curvature function (and thus by its geodesic curvature function). Then recalling a result due to Kappeler [22], we note that finite gap $L^2(\mathbb{S}^1, \mathbb{R})$ functions are dense in the set of all $L^2(\mathbb{S}^1, \mathbb{R})$ functions. As a result of this the set of curves generated by finite gap geodesic curvature arising in this way, are dense in the set of all curves on $\mathbb{S}^2$ whose geodesic curvature functions lie in $L^2(\mathbb{S}^1, \mathbb{R})$. We then prove that finite type flat surfaces are dense in the set of all flat surfaces in $\mathbb{S}^3$ that are generated by admissible pairs with geodesic curvature functions in $L^2(\mathbb{S}^1, \mathbb{R})$.

6.1 Curves on $\mathbb{S}^2$

Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a regular curve that satisfies $\tau \neq 0, \kappa \neq 0$.

Proposition 6.1.1. *[7] γ lies on a sphere if and only if τ and κ satisfy*

$$\frac{\tau}{\kappa} = \frac{d}{dt}\left(\frac{\kappa'}{\tau\kappa^2}\right).$$

Corollary 6.1.2. *Let γ be a curve on $\mathbb{S}(p, r)$, the sphere of radius r and centre p, with curvature κ and torsion τ. Then*

$$\tau = \frac{\kappa'}{\kappa\sqrt{r^2\kappa^2 - 1}}.$$

Proof. Differentiating $\|\gamma(t) - p\| = r$ twice yields $\langle T', \gamma - p\rangle = -1$ and thus it follows that $\kappa(t) = \|T'(t)\| \neq 0$. Hence N is well defined by $T' = \kappa N$. Rewriting yields $\kappa\langle N, \gamma - p\rangle + 1 = 0$ and differentiating:

$$\begin{aligned} 0 &= \kappa'\langle N, \gamma - p\rangle + \kappa\langle -\kappa T + \tau B, \gamma - p\rangle + \kappa\langle N, T\rangle \\ &= \kappa'\langle \frac{1}{\kappa}T', \gamma - p\rangle + \kappa\tau\langle B, \gamma - p\rangle \\ &= -\frac{\kappa'}{\kappa} + \kappa\tau\langle B, \gamma - p\rangle. \end{aligned}$$

Now $\{T, N, B\}$ is an orthonormal basis and hence

$$\begin{aligned} r^2 &= \|\gamma - p\|^2 \\ &= \langle \gamma - p, T\rangle^2 + \langle \gamma - p, B\rangle^2 + \langle \gamma - p, N\rangle^2 \\ &= 0 + \frac{1}{\kappa^2} + \langle \gamma - p, B\rangle^2. \end{aligned}$$

Combining both results it follows that

$$-\frac{\kappa'}{\kappa} + \kappa\tau\sqrt{r^2 - \frac{1}{\kappa^2}} = 0.$$

Hence

$$-\frac{\kappa'}{\kappa} + \kappa\tau\sqrt{\frac{r^2\kappa^2 - 1}{\kappa^2}} = 0,$$

which implies that

$$\tau = \frac{\kappa'}{\kappa\sqrt{r^2\kappa^2 - 1}}.$$

□

Thus as a result of this we have that curves on $\mathbb{S}^2$ are determined by their geodesic curvature function.

Definition 6.1.3. 1. Let $L^2(\mathbb{S}^1, \mathbb{R})$ be the space of real valued periodic functions with period 2π. Then define

$$\begin{aligned} \mathcal{H}^N(\mathbb{S}^1, \mathbb{R}) &= \{u \in L^2(\mathbb{S}^1, \mathbb{R}) : D^\alpha u \in L^2(\mathbb{S}^1, \mathbb{R}), |\alpha| \leq N\} \\ \mathcal{H}_0^N(\mathbb{S}^1, \mathbb{R}) &= \{u \in L^2(\mathbb{S}^1, \mathbb{R}) : D^\alpha u \in L^2(\mathbb{S}^1, \mathbb{R}), |\alpha| \leq N,\ [u] = 0\} \end{aligned}$$

where $[u] = \int_0^{2\pi} u(t)dt$ and $D^\alpha u$ is the weak partial derivative of order α of u. The norm on $\mathcal{H}^N$ is given by

$$\|u\|_N^2 = |\hat{u}(0)|^2 + \sum_{k\in\mathbb{Z}^*} |k|^{2N}|\hat{u}(k)|$$

where

$$\hat{u}(k) = \int_0^{2\pi} u(x)e^{-2\pi i x k}dx.$$

2. The KdV equation is

$$KdV(V) = V_t - 6VV_x + V_{xxx} = 0.$$

The following well known result is stated [22, p.204]:
Theorem 6.1.4. *Finite gap potentials of KdV are dense in $\mathcal{H}_0^N(\mathbb{S}^1, \mathbb{R})$.*

Note. Note that $\{\mathcal{H}_c^N\}_{c\in\mathbb{R}}$ forms a foliation of $\mathcal{H}^N$ where all the leaves (given by $[u] = c$, c a constant) are symplectomorphic. Thus finite gap potentials of KdV are dense in $\mathcal{H}^N$.

Remark. Define the Miura transformation by $p(u(x)) = u(x)^2 - u'(x)$. This transformation takes solutions of mKdV to solutions of KdV and this process can be reversed. In particular all periodic finite gap solutions of mKdV are generated from periodic finite gap solutions of KdV (see [9], [11], [12], [13], or [14]) and we note that conversely, all periodic finite gap solutions of KdV are generated from periodic finite gap mKdV solutions using the Miura transformation.

Moreover since p is continuous we have
Corollary 6.1.5. *Finite gap (mKdV) solutions are dense in the set of C^1 functions in $\mathcal{H}_0^N(\mathbb{S}^1, \mathbb{R})$, $N \geq 1$.*

Proof. Assume to the contrary that finite gap solutions of mKdV are not dense in $\mathcal{H}_0^N(\mathbb{S}^1, \mathbb{R})$. Then there exists a C^1 function $u \in \mathcal{H}_0^N(\mathbb{S}^1, \mathbb{R})$ and $\delta > 0$ such that the open ball $B(u, \delta)$ contains no finite gap solutions of mKdV. Let p be the Miura transformation and choose $\epsilon > 0$ such that $p^{-1}(B(p(u), \epsilon)) \subset B(u, \delta)$. Then since finite gap solutions of KdV are dense in $\mathcal{H}_0^N(\mathbb{S}^1, \mathbb{R})$, there exists $v \in B(p(u), \epsilon)$ such that v is a finite gap solution of KdV. Moreover since every finite gap solution of mKdV is generated from a finite gap solution of the Miura transformation and its reversal, we have that there exists a finite gap solution of mKdV $w \in B(u, \delta)$ such that $p(w) = v$. This gives the required contradiction. □

As a result of Theorem 6.1.4, corollary 6.1.2 and corollary 6.1.5, it follows that:

Lemma 6.1.6. *Finite gap curves on $\mathbb{S}^2$ generated by finite gap solutions to mKdV in $\mathcal{H}_0^N(\mathbb{S}^1,\mathbb{R})$ are dense in the set of all curves on $\mathbb{S}^2$ with C^1 geodesic curvature functions in $\mathcal{H}_0^N(\mathbb{S}^1,\mathbb{R})$.*

Let A be the set of all immersed flat surfaces in $\mathbb{S}^3$ whose admissible pairs have C^1 geodesic curvature functions in $\mathcal{H}_0^N(\mathbb{S}^1,\mathbb{R})$. Then following Kitagawa's classification [23] we have:

Theorem 6.1.7. *Flat surfaces of finite type are dense in A.*

Proof. Let $\Gamma = (\gamma_1, \gamma_2)$ be an admissible pair with corresponding curvatures κ_1, κ_2 and geodesic curvatures $(\kappa_g)_1, (\kappa_g)_2$. Then the curves satisfy

(1) $\gamma_i(0) = e_3, \dfrac{\gamma_i'(0)}{\|\gamma_i'(0)\|} = e_1$

(2) $\|\gamma_i'\|^2(1 + (\kappa_g)_i^2) = 4$

(3) $(\kappa_g)_1(t_1) > (\kappa_g)_2(t_2)$ for all $(t_1, t_2) \in \mathbb{R}^2$.

Set $\epsilon = min(|(\kappa_g)_1(x) - (\kappa_g)_2(y)|)$, $x, y \in \mathbb{S}^1$ which is bigger than 0 by (3). Let $\int_0^{2\pi}(\kappa_g)_1 dx = c$. Then there exists a finite gap function κ_{F1} in $\mathcal{H}_c^1$ such that

$$\|(\kappa_g)_1(x) - \kappa_{F1}(x)\| \tag{6.1}$$

$$= \sqrt{|(\kappa_g)_1\widehat{(x) - \kappa}_{F1}(x)(0)|^2 + \sum_{k\in\mathbb{Z}^*}|k|^{2N}|(\kappa_g)_1\widehat{(x) - \kappa}_{F1}(x)(k)|} \tag{6.2}$$

$$< \frac{\epsilon\sqrt{3}}{\pi}. \tag{6.3}$$

Then recalling the inverse Fourier transform

$$u(x) = \sum_{j\in\mathbb{Z}} e^{2\pi ijx}\hat{u}(j)$$

and noting that since $(\kappa_g)_1$ and κ_{F1} are both elements of $\mathcal{H}_c^1$ we have that the

Fourier transform $(\widehat{(\kappa_g)_1 - \kappa_{F1}})(0) = 0$. Then

$$\begin{aligned}
|(\kappa_g)_1(x) - \kappa_{F1}(x)| &= \left| \sum_{j\in\mathbb{Z}^*} e^{2\pi ijx} (\widehat{(\kappa_g)_1 - \kappa_{F1}})(j) \right| \\
&\leq \sum_{j\in\mathbb{Z}^*} |(\widehat{(\kappa_g)_1 - \kappa_{F1}})(j)| \\
&= \sum_{j\in\mathbb{Z}^*} \frac{1}{j} j |(\widehat{(\kappa_g)_1 - \kappa_{F1}})(j)| \\
&\leq \sqrt{\sum_{j\in\mathbb{Z}^*} \frac{1}{j^2}} \sqrt{\sum_{j\in\mathbb{Z}^*} j^2 |(\widehat{(\kappa_g)_1 - \kappa_{F1}})(j)|} \\
&< \frac{\pi}{\sqrt{3}} \frac{\epsilon\sqrt{3}}{\pi} \\
&= \epsilon
\end{aligned}$$

where the second inequality is an application of Cauchy-Schwarz and the third inequality follows from (6.1). Similarly there exists a finite gap function κ_{F2} such that $|\kappa_2(x) - \kappa_{F2}(x)| < \epsilon$. Hence $\kappa_{F1}(x) > \kappa_{F2}(y)$ for all $x, y \in \mathbb{S}^1$. Thus the curves α_1, α_2 on $\mathbb{S}^2$ generated by κ_{F_1} and κ_{F_2} satisfy (3). Moreover with a suitable isomorphism and reparametrization they can be made to satisfy (1) and (2) and are thus an admissible pair. □

Remark. Furthermore, if κ_F is a finite gap geodesic curvature function whose resultant curve is closed (that is, if the monodromy of its frame is $\pm\mathbb{1}$ at the sympoint), then by proposition 5.7.2 the generated surface will be a torus

6.2 Open Problems

One remaining open question is to whether flat tori of finite type (that is flat tori generated by curves on $\mathbb{S}^2$ whose geodesic curvature function is finite gap) are dense in the set of all flat tori. While this question remains open we do have the following result due to Nikolaevsky [25] which suggests that there is no effective way to ensure closure from just the curvature function.

Theorem 6.2.1. *Let $F(x, y)$ be a continuous function and denote by $C^2(l)$ the set of all C^2-regular closed curves on $\mathbb{S}^2$ of length l. Then either*

1. *the set of values of the integral*

$$\int_0^l F(\kappa(s), s)$$

computed for all curves in $C^2(l)$ contains an interval or

2. *$F(x, y) = \phi(y)$ where $\phi(y)$ is a periodic function with period l.*

In the second case the value of the integral is not dependent on the geodesic curvature and as such the set of such curves is empty.
Problems that present themselves for future investigation include:
Closure of finite gap curves: While theorem 6.2.1 suggests that we cannot ensure closure by placing additional restrictions on the curvature, an approach to overcome this in order to study Hopf tori and flat tori in general, is to use isoperiodic deformations to force the finite gap curve to close up by choice of a suitable deformation in order to ensure that the extrinsic closing conditions are attained. In fact we conjecture that finite type flat tori are dense in the set of all flat tori.
Classification of the moduli space of flat tori of finite type: One avenue of potential new results is to study the moduli space of flat tori of finite type using isospectral deformations. In particular it may be possible to show that the moduli space consists of two connected components generated by the class of curves deformable to a single wrapped circle and the class of curves deformable to a twice wrapped circle.
Flowing through flat tori of finite type: Firstly we note that for a flat surface (and in particular torus) of finite type, we can define its spectral curve to be

$$\Sigma_{12} = \{(\Sigma_1,\, \Sigma_2)\}$$

where Σ_1 and Σ_2 are the spectral curves of the admissible pairs that generate the flat torus, along with a pair of functions $(\mathrm{dlog}(\mu_1),\ \mathrm{dlog}(\mu_2))$ that satisfy

$$\frac{\mathrm{d}}{\mathrm{dt}}\log(\mu_i)|_{\lambda_{sym}} = 0 \text{ and } \quad \mathrm{dlog}(\mu_i)|_{\lambda_{sym}} = 0.$$

Then a further problem is to establish the relationship between the eigenvalue of the monodromy μ and the geodesic curvatures of the curves associated to the spectral curve in order to determine how to flow periodic admissible pairs in such a way as to keep them admissible. To do this we will require that the evolving curves maintain a distance between their respective geodesic curvatures in order to remain admissible. Potential approaches to solving this problem include considering flows that only increase the larger geodesic curvature or decrease the smaller geodesic curvature while also satisfying the necessary closing conditions. Another approach is to investigate short time existence of flows that do not allow the different geodesic curvatures to become equal or to use a different flow such as the curve shortening flow.

Investigation of finite gap curves in $\mathbb{S}^3$: Following Bianchi-Spivak we may also generate flat surfaces in $\mathbb{S}^3$ by looking directly for curves in $\mathbb{S}^3$ with torsion equal to ± 1, bypassing the need for lifts using the Hopf fibration. It would then be interesting to study the finite type flat surfaces generated by such finite gap curves and to investigate the correspondence between finite gap admissible pairs, finite gap asymptotic curves in $\mathbb{S}^3$ and their corresponding spectral curves. Indeed following Grinevich [16], it can be shown that finite gap curves generated by periodic complex curvature functions q are dense in the set of all curves in $\mathbb{S}^3$ with periodic q, so the open question remains as to whether these always correspond to a lift of a finite gap curve in $\mathbb{S}^2$ or not and how to classify those finite gap curves in $\mathbb{S}^3$ that are asymptotic.

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